EMILE PICARD, 1856–1941

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Since the last quarter of the eighteenth century mathematicians have appeared in France in groups. The first of these would include such figures as Lagrange, Monge, Laplace, Legendre, Fourier, and Cauchy.

Picard was the last survivor of an epoch which rose to its apogee with the researches of Poincaré and was initiated by Hermite. These men might well have chosen as their symbol the modular function. Hermite introduced it, and for Poincaré it constituted the beginning of an admirable chapter in mathematics—the theory of automorphic functions. It was given to Picard to make its most impressive application in the theory of functions.

The proofs of both theorems of Picard on entire functions: "the first theorem" and "the great theorem"—both great—are so astonishing that one need not hesitate to describe them as dramatic.

Generalizations of these theorems (Borel, Landau, Schottky, Carathéodory, Julia, Bloch), theories connected with them (normal families of Montel, characteristic functions of Nevanlinna) and their applications are so numerous and important, that scores of well-known mathematicians have devoted and still devote their lives to them. The results obtained constitute a large autonomic field comparable in importance only to the great classical domains of modern mathematics.

In relation to Picard's theorems on entire functions (or more generally on the behavior of analytic functions in the neighborhood of isolated singularities) one should remember the rôle, in the theory of discontinuous groups, of the "group of Picard."

Followers of Picard attempted to avoid the use of the modular function in the proofs of his theorems, in an effort to provide an "elementary proof" of them. Certain important results were obtained in this direction, but it is remarkable that other not less important contributions have demonstrated that this function is intrinsically related to the field created by Picard.

It should be pointed out, perhaps, that the idea of the Riemann surface appears in its true light through the topological aspects of Picard's theorem.

A capital contribution to the understanding of general existence theorems in differential equations is given by Picard's method of successive approximations. It furnished its author, and many of his followers, with a speedy proof of convergence of a series which gives the integral, replacing advantageously the older methods of Cauchy-Lipschitz. Its constructive character permits a new outlook on questions of existence, and, in many ways has proved to be of great utility.

We owe to Picard the development of the theory of algebraic functions of many variables, a study which was originated by Riemann. In this connection should be mentioned Picard's work on algebraic surfaces with rational plane sections.

It is of course impossible to do more than mention Picard's work on partial differential equations, for instance, that concerning the methods of integration of equations of hyperbolic type (related to the propagation of electrical waves through a homogeneous telegraphic wire), or methods employed in various forms of the solution of the Dirichlet problem.

The "method of Schwartz-Picard-Poincaré" in the theory of integral equations will recall Picard's interest in this subject which was, for many years, his favorite field of investigation and instruction.

Theory of functions, theory of groups, algebraic functions of many variables, theoretical physics, differential equations were perhaps the branches of mathematics which Picard enriched most, but he was not one of those whom one may describe as "a specialist" in one or another field—he was one of the few Masters who formed the very basis of our mathematical science.

There is not a single French mathematician over thirty years of age who was not a student of his. But not only French students enjoyed his Cours d'Analyse Supérieure. The Amphitheatre at the Sorbonne where be taught for 50 years was the meeting place of young mathematicians from all over the world. Even those who had some difficulties in understanding French could admire the simplicity and directness of his teaching. Sometimes, what is generally called "an elegant exposition" seems unpleasant in mathematics, since elegance is often won by hiding the real difficulties of the subject. Is it that simplicity was inherent in the subjects he used to teach, or is it that he made these subjects susceptible of simplification? At any rate students could but admire the Master as well as the subject. One can form an idea of his manner of teaching by reading his famous "Traité d'Analyse."

Picard has had, of course, all the distinctions a great mathematician can have: membership in French and foreign Academies, Scientific Societies, etc. But in France his prestige surpassed even that of some of the greatest representatives of our science. His position in the mathematical world, his profound interest in the history of French thought, exemplified in the beautifully written essay on Pascal, have made him recognized representative of French Science. And if the official consecration of his prestige was his election as Secrétaire Perpétuel de l'Académie des Sciences, and—a rare honor for a scientist—as "Immortel," Member of the Académie Française, his olympian attitude reflected a sense of the rôle he played in his time. He was a worthy representative of a proud epoch.

ANNUAL MEETING OF NORTHERN CALIFORNIA SECTION

The fourth annual meeting of the Northern California Section was held at the University of California, Berkeley, on Saturday, January 31, 1942. Professor F. R. Morris, chairman of the Section, presided at both morning and afternoon sessions. During the noon recess luncheon was served at the Men's Faculty Club.

The attendance at the two sessions was sixty-five, including the following twenty members of the Association: H. M. Bacon, G. A. Baker, T. J. Bass, B. A. Bernstein, G. C. Evans, Emma Hesse, Einar Hille, D. H. Lehmer, Sophia H. Levy, A. L. McCarty, E. D. Miller, F. R. Morris, W. H. Myers, E. B. Roessler, Ethel Spearman, Pauline Sperry, Ruth G. Sumner, Gabor Szegö, A. R. Williams, Fredrick Wood.

The following officers were elected for the coming year: Chairman, Fredrick Wood, University of Nevada; Vice-Chairman, E. B. Roessler, University of California at Davis; Secretary-Treasurer, H. M. Bacon, Stanford University. Mrs. Ruth G. Sumner, Oakland High School, was re-elected to represent the Section as associate editor of the *California Journal of Secondary Education.* Professor Sophia H. Levy reported briefly on the activities of the Committee on Mathematical Education of the two California Sections which had been authorized by a mail vote of the two Sections and appointed by the Regional Governor; the membership of the Committee is as follows: Sophia H. Levy, Chairman; C. G. Jaeger, E. B. Roessler, S. E. Urner, W. M. Whyburn, H. M. Bacon (ex-officio), G. C. Evans (ex-officio).

By invitation of the Section, Professor H. C. Burbridge of Fresno State College gave an hour's address during the morning session.

The following papers were read:

1. "Theory of budgets based on parabolic Engel curves" by Dr. G. A. Baker, University of California at Davis.

2. "Concerning the altitudes of a tetrahedron" by Professor J. H. McDonald, University of California, introduced by Professor Levy.

3. "Recent developments in engineering" by Professor H. C. Burbridge, Fresno State College, by invitation.

4. "Senior mathematics: a semester course in algebra and geometry designed for students without previous high school mathematics" by L. J. Hill, San Jose High School, introduced by Professor Myers.

5. "On conjugate trigonometric polynomials" by Professor Gabor Szegö, Stanford University.

6. "On unbounded solutions of linear second order differential equations" by Professor Einar Hille, Yale University.

Abstracts of the papers follow, numbered in accordance with their listing:

1. Dr. Baker showed that, considering averages, expenditures on groups of items are parabolic functions of the total income or expenditure. If total income is divided into expenditure on savings and expenditures on all other

items, then the preference curve is parabolic and a quadratic utility surface is determined except for an additive constant. The indifference curves are arcs of ellipses which are orthogonal to the preference curve.

2. Professor McDonald showed that the altitudes of a tetrahedron in general are four lines of a regulus; the quadric of this regulus contains eight other lines. These are the four perpendiculars to the faces of the tetrahedron erected at their orthocenters, and four other lines. Any one of the latter is determined as the line of intersection of three planes drawn through three concurrent edges of the tetrahedron and perpendicular, in each case, to the face of the other two edges. It may be shown that the quadric is a rectangular one, namely, such that when given by an equation in rectangular coordinates the sum of the coefficients of the squares is zero, and characterized geometrically by the property that each of its reguli contains sets of three mutually perpendicular lines.

3. Professor Burbridge described developments in aviation including testing devices, design of planes, development of fuels, methods of testing the smoothness of surfaces and their relation to the design and construction of machines. A particularly interesting illustration described the securing of the same result by two widely different methods of research. Several developments in the fields of radio and telephony were discussed, especially the increase in the number of telephone meassages which can be carried by a single cable. The role of scientists in saving England during the present war was mentioned, together with a brief account of recent developments in astronomy.

4. At the present time many senior students in San Jose High School wish to change from fields requiring little or no mathematics to fields requiring a mathematical background. Mr. Hill described a semester course to cover briefly the fundamentals of algebra, geometry, and plane trigonometry which would attempt to meet as well as possible the needs of these senior students in the very limited time available.

5. Professor Szegö considered the set of all polynomials $f(z)$ of degree n whose real part lies between preassigned bounds, say between -1 and $+1$ as $|z| \leqq 1$; let $f(0)$ be real. The problem is to find the greatest possible value for the imaginary part of $f(z)$, $|z| \leqq 1$. After transforming this problem into one in terms of conjugate trigonometric polynomials the solution follows by means of a proper interpolation formula. The bound in question is $\cong(2/\pi) \log n$ as $n \to \infty$, attained for certain polynomials having the form $\pm f_0(\epsilon z)$, $|\epsilon| = 1$; here $f_0(z)$ is a special polynomial of the given set with pure imaginary coefficients which assumes the values $+1$ and -1 at the points $m\pi/(n+1)$ and $-m\pi/(n+1)$, respectively, $1 \leqq m \leqq n$, m odd. These are the only extremum polynomials if n is odd, whereas for even n besides these some other extremum polynomials exist.

6. Professor Hille assumed the given linear second order differential equation to be self-adjoint and the solutions to be non-oscillatory in an interval (a, b). A necessary and sufficient condition was given in order that the solutions be unbounded in (a, b). It was shown by examples that the inequalities obtained are the best possible.

H. M. BACON, *Secretary*

WHAT IS THE JORDAN CURVE THEOREM?

J. R. KLINE, University of Pennsylvania

In non-technical terms we may define a simple closed curve or Jordan curve as the most general set which can be obtained from a circle by bending and stretching without breaking or crossing. More precisely, a simple closed curve is the image of a circle under a homeomorphism, *i.e.*, under a (1–1) continuous transformation with a continuous inverse, while a simple continuous arc is the image of a straight line interval under a transformation of the same type. While such simple figures as a triangle or an ellipse are simple closed curves, we can also obtain simple closed curves the structure of which is extremely complex. We shall exhibit two curves of the latter type to show that, under a homeomorphism, a circle does not necessarily retain any one of the following properties:

(a) that of having a tangent at every point;
(b) that of having a length;
(c) that of having its two-dimensional area (measure) equal to zero.

That (a) and (b) may be lost can easily be seen from the following well-known simple closed curve: let C_0 be an equilateral triangle while C_1 is the outer boundary of the figure obtained if, on the middle third of each straight line segment of C_0, we erect an equilateral triangle whose interior lies wholly without C_0. Assuming that C_n has been constructed, we obtain C_{n+1} as the outer boundary of the figure reached by erecting, on the middle third of each straight line segment of C_n, an equilateral triangle whose interior lies wholly in the exterior of C_n. Let C be the limit of C_0, C_1, $C_2 \cdots$. This limit is a simple closed curve. Clearly at no point does C possess a tangent while the length of the perimeter of C_n becomes infinite with n.

To show that property (c) is not necessarily retained, we have recourse to a scheme based on the well-known Cantor middle third process. The Cantor process provides us with a nowhere dense perfect set on a straight line interval. Our modification leads to a perfect set C which contains no connected subsets other than single points and has plane measure different from zero.* Now with the use of a theorem due to R. L. Moore and the author [1] we are able to construct a simple closed curve J which contains every point of C and hence must

* In this footnote we shall describe the process by which we obtain C. Let C_0 denote the square $ABCD$ plus its interior. To obtain C_1, we remove from C_0 all of those points which lie between two perpendiculars to BC which (1) are symmetric to the perpendicular bisector of BC and (2) include between them 1/3 of the square.

Next, remove from C_1 that portion of C_1 which lies between two lines which (1) are perpendicular to AB and are symmetric to the perpendicular bisector of AB and (2) are such that the parts of the two rectangles, which comprise C_1, included between them, have together an area equal to 1/9 the original square. Call the four remaining rectangles C_2.

Next on *each* of the intervals of BC which remain erect a pair of lines which are (1) perpendicular to BC and symmetric to the perpendicular bisector of the interval and (2) are such that

have plane area different from zero. These examples contradict many of our more intuitive notions of a curve.

In view of these examples, one naturally wonders what properties, if any, of the circle C must remain after it has been subjected to so general a transformation as a homeomorphism. We do find that our conditions will be sufficient to require the resulting set J still to possess properties such as (1) that of having any two distinct points separate J into two mutually separated connected pieces of which they are the common boundary and (2) that of remaining connected after the removal of any connected piece. Both of these properties are, of course, possessed by the original circle. Much more interesting, however, is the relation of the resulting simple closed curve J to the plane S in which it may be imbedded. From elementary geometry we remember that a circle divides the remaining points of the plane into two sets, the interior and the exterior, the former consisting of points at a distance less than the radius from the center, while the latter contains the points whose distance from the center is greater than the radius. These sets have the property that, in order to join a point of one set to a point of the other by any simple continuous arc, one must pass through a point of the circle. Intuitively one feels certain that a separation of the remaining points into distinct sets must still persist after the circle has been carried by a homeomorphism into any simple closed curve in the plane. Before the introduction of modern standards of mathematical rigor, mathematicians generally accepted the existence of such a separation without proof. However, in 1865 C. Neumann in his *Vorlesungen über die Riemannschen Theorie der Abelschen Integrale* explicitly asked for a proof of the existence of such a division of the plane.

In 1887, C. Jordan gave the first so-called proof of the theorem that now bears his name, *i.e., if a simple closed curve J is imbedded in a plane S, then $S-J$ is the sum of exactly two mutually exclusive domains and every point of J is a boundary point of each domain.* Jordan's argument did not suffice even for the case of a polygon. The first complete proof of the theorem in its most general form was given by Veblen [2] in the *Transactions of the American Mathematical Society* in 1905. Subsequently many other proofs were given. Prominent among

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the parts of the rectangles of C_2 included between the two pairs of perpendiculars have together an area equal to 1/27 of the whole square. Call the eight rectangles which remain C_3.

Next we turn to the invervals of AB which remain and perform a similar construction, removing in all from C_3 an amount whose area is 1/81 of the original square. We proceed in this manner, removing alternately strips symmetric to the perpendicular bisectors of all the intervals of BC that survive, and then strips symmetric about perpendicular bisectors of all intervals of AB that survive. We arrange the width of the strips in such a manner that at the end of the nth step we remove from C_{n-1}, a total area equal to $(1/3)^n$ of the original square.

Let C be the set of points common to $C_0, C_1, C_2, \cdots$. It is clear that this set is (1) closed (2) contains no connected subset other than a single point. As the part of the square which has been removed has an area equal to 1/2 of the square, it is clear that C has positive measure equal to 1/2 the square.

By a suitable modification of this process we can obtain a totally disconnected closed set whose area is any *proper* fraction of the area of the square.

these are the proofs by Brouwer, Alexander, Kerekjarto, Schoenflies, and Winternitz. A very satisfactory recent proof is that contained in the chapter on separation theorems in a recent book by M. H. A. Newman [3]. The principal steps in the proof are the proving of (1) the fact *that a simple continuous arc does not separate the plane* S and (2) the fact that if F_1 and F_2 are two bounded continua, neither of which separates S, then a necessary and sufficient condition that F_1+F_2 shall separate S is that the common part $F_1 \cdot F_2$ be not connected.

One naturally inquires as to the truth of the converse of the Jordan Curve Theorem. If we apply the term "closed curve" to any closed and bounded plane set which divides its plane into exactly two parts of which it is the common boundary, then our question is whether every closed curve is necessarily a simple closed curve. That this is not true may easily be seen by considering the example which is so frequently used by topologists, *i.e.*, the set composed of (1) $y=\sin\pi/x$ $(0<x\leqq 1)$, (2) the interval of the y-axis from $(0, -2)$ to $(0, 1)$, (3) the interval of $y=-2$ from $(0, -2)$ to $(1, -2)$, and (4) the interval of $x=1$ from $(1, -2)$ to $(1, 0)$. We mention three conditions which are such that, if any one of them is added to the hypothesis that our set C is a *closed curve*, our set then becomes a simple closed curve; and conversely each simple closed curve in a plane has these properties. They are:

(1) Every point of C is accessible from both of the complementary domains $D_i(i=1, 2)$ where a point P of C is said to be accessible from D_i, if, for every point Q of D_i, there exists a simple continuous arc from Q to P which lies, except for P, entirely in D_i. This is the condition which was used by Schoenflies [4] in obtaining a proof of the converse of the Jordan Curve Theorem.

(2) The set C is locally connected at every point. This condition was exhibited by the author [5] of this paper as a necessary and sufficient condition that a closed curve be a simple closed curve.

(3) The set C at each of its points separates the plane locally into the same finite number, n, of domains. The plane is cut locally at P into n parts by C if, for every sufficiently small neighborhood U_P of P, the set U_P-C has at least n components and there exist arbitrarily small neighborhoods for which the number of components is exactly n. This is the condition of Zarankiewicz [6].

The simple closed curve J is obtained from the circle by a homeomorphic transformation π which is concerned only with the points of the circle C and their image points on J. Nothing whatsoever is assumed as to what happens to the remaining points of the plane S. There is, however, a powerful theorem, due originally to Schoenflies [7], according to which whenever we have been given π, then we can define a new homeomorphism π' of the entire plane S into itself which has the property that, insofar as points of C are concerned, their image under π' is exactly the same as their image under the original π. When we turn to plane figures which are but slightly more complicated, such an extension of the homeomorphism is no longer always possible. Consider, for instance, the set consisting of three simple continuous arcs AB_iC $(i=1, 2, 3)$ in which (1) no two arcs have a point, other than A and C, in common and (2) except for its

endpoints AB_2C lies wholly in the interior of the simple closed curve AB_1CB_3A. Let π be a homeomorphism such that A and C remain fixed and $\pi(AB_iC) = AB_{i+1}C$ when subscripts are reduced modulo 3. Clearly this homeomorphism cannot be extended, for, were the extension possible, the exterior of AB_1CB_3A would necessarily be transformed into the interior of AB_1CB_3A under the extended transformation (Figure 1). Much interesting research has been done on the problem of when a transformation of a subset of the plane can be extended to a transformation of the entire plane into itself.

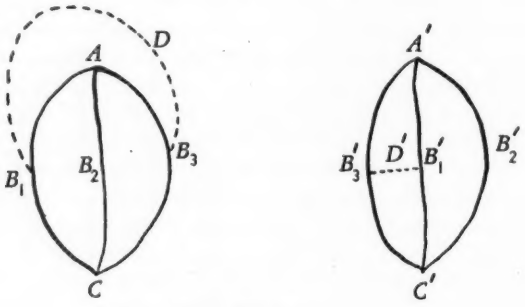

Fig. 1

When we turn to three or more dimensions, the situation becomes more interesting and also much more complicated. That the homeomorph of a Euclidean n-sphere divides Euclidean space of $(n+1)$ dimensions was first proved by Brouwer [8]. In 1924 Alexander [9] gave an example of a simple closed surface S, *i.e.*, the homeomorph of the sphere $x^2+y^2+z^2=r^2$ which has in its exterior a simple closed curve which cannot be shrunk to a point without cutting S. Of course every simple closed curve in the exterior of $x^2+y^2+z^2=r^2$ can be shrunk to a point while remaining entirely in the exterior. By virtue of this example, we see immediately that a theorem analogous to Schoenflies' extension theorem is not true for simple closed surfaces in space of three dimensions.

In addition, in space of three or more dimensions, there are offered for our consideration sets much more general than the homeomorph of a sphere of one less dimension. As an illustration in three dimensions, we take the torus T with the system of meridianal and longitudinal simple closed circles thereon. When the torus is removed from three space, not only do we have a pair of points which cannot be joined by an arc without crossing T, but we have also a pair of simple closed curves, one in the bounded domain cut off by T and the other in the unbounded domain, neither of which is the boundary of a two-cell which does not cut T. The number of independent non-bounding simple closed curves in E_3-T is and must be the same as the number of independent non-bounding simple closed curves on T. This is a very simple case of the consequences of

Alexander's Duality Theorem [10]; this powerful theorem was published in 1922. In contains the Jordan Curve Theorem as a corollary. The duality theorem has been the subject of a long series of important investigations culminating in Pontrjagin's general topological theorem of duality for closed sets [11].

The Jordan Curve Theorem and ideas connected therewith play an important role in R. L. Moore's work on the *Foundations of Plane Analysis Situs* [12] and in the subsequent far-reaching research which he and his students have done. While it is impossible even to indicate all the principal directions that this theorem has given to research in Topology, I wish to call attention to two of its important applications. In the first place, it has been of fundamental importance in determining those continuous curves which are such that on the surface of a sphere there can be found a set which is the homeomorphic image of the original curve. Here fundamental work has been done by Kuratowski and Claytor [13]. The Jordan Curve Theorem also plays a very important role in setting up necessary and sufficient conditions which must be satisfied in order that a continuum shall be a simple closed surface. Here Zippin [14] proved that any compact locally-connected continuum which satisfies the Jordan Curve Theorem *non-vacuously* must be a simple closed surface. Other interesting work in applying the Jordan Curve Theorem to the definition of simple closed surfaces and two-dimensional manifolds has been done by Miss Gawehen, van Kampen [15], J. H. Roberts, Hassler Whitney, and R. L. Wilder. If, in addition to the direct applications of the Jordan Curve Theorem, we consider also the research which is based on the ideas and methods developed in connection with it and its various extensions, we find that it has had an influence on the mathematics of the past fifty years which is surpassed by few, if any, other theorems.

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REMARKS ON DIVISORS OF ZERO*

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1. Introduction. This paper deals with certain elements of those important algebraic systems which are called *rings*. For the sake of completeness, we shall first give definitions of some of the fundamental concepts with which we shall be concerned.†

A *ring* R is a set of elements $a, b, c, \cdots$, with the property that for any two elements a and b of R there is a uniquely defined sum $a+b$ and product ab which are elements of R, and such that the following are always true:

$$\begin{aligned} a + (b + c) &= (a + b) + c, \\ a + b &= b + a, \\ a + x &= b \text{ has a solution } x \text{ in } R, \\ a(bc) &= (ab)c, \\ a(b + c) &= ab + ac, \\ (b + c)a &= ba + ca. \end{aligned}$$

If, for every a and b in R, $ab=ba$, the ring R is said to be a *commutative* ring, otherwise it is a *non-commutative* ring. We shall, for the most part, be concerned with commutative rings.

The above properties are certainly satisfied by all the number systems of elementary algebra with the ordinary definitions of sum and product. Thus, as

* An address delivered at the Bethlehem meeting of the Mathematical Association of America on January 1, 1942.

† The terms and ideas introduced in this section are to be found in van der Waerden [11], and also in most other texts on abstract algebra.

simplest examples, we have the ring of real numbers, the ring of rational numbers or the ring of integers. Another illustration of this concept, and one of a different nature, is the so-called *ring of integers modulo* 6, that is a ring of six elements 0, 1, 2, 3, 4, 5 with $a+b$ defined as the least non-negative remainder when the ordinary sum of a and b is divided by 6, and a similar definition of products. Thus, for example, in this ring we have $4+5=3$ and $2\cdot 4=2$. Naturally, any positive integer m may be used in place of the 6 in this example. Throughout, the ring of integers modulo m will be denoted by I_m.

All the rings introduced so far are commutative rings. As a familiar example of a non-commutative ring, consider the set of all matrices of order two with elements in a ring R, with addition and multiplication defined in the usual way as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix},$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

It is easily verified that the set of all such matrices is a non-commutative ring, even if R is commutative, which we may denote by R_2. In like manner, the ring R_n may be used to denote the ring of all matrices of order n with elements in R.

From the definition of a ring, it is easy to prove that there is always a unique element 0 with the property that, for all a in R, $a+0=a$. This element is naturally called the *zero* of R. It is quite easy to show, as in ordinary algebra, that $a0=0a=0$ for all a in R. However, in the ring I_6, we see that $2\cdot 3=0$. Thus it may happen, in a ring, that a product is zero without either factor being zero. We thus come to the following definition. An element a of a ring R is a *divisor of zero* in R if there exists a non-zero element b of R such that $ab=0$, or a non-zero element c of R such that $ca=0$. If $ab=ba=0$, $b\neq 0$, we may say also that b is an *annihilator* of a, or that a is *annihilated* by b. To avoid confusion later on, we may emphasize that an annihilator is necessarily different from zero.

According to the definition, zero is always a divisor of zero, but it will be convenient and also cause no misunderstanding if, instead of saying that a ring has no divisors of zero except zero, we merely say that a ring has no divisors of zero. Thus, for example, the ring of rational numbers clearly has no divisors of zero.

In all the examples of rings mentioned so far there is an element 1 with the property that $1\cdot a=a\cdot 1=a$ for every element a of the ring. Such an element is called a *unit element* of the ring. We shall assume throughout that all rings considered have unit elements.

Now the possible presence of divisors of zero in a ring accounts for a large part of the differences which arise in algebraic manipulation of elements of a ring, at least in the commutative case, as compared with operations on ordinary numbers. In this paper, we shall present several miscellaneous theorems having to do with divisors of zero. Inasmuch as any calculation with elements of a ring is likely to lead to considerations of divisors of zero, the literature is extensive and the few topics presented here are to be considered merely as a sample and not as an exhaustive treatment of the subject.

2. Linear homogeneous equations. Let us begin by a consideration of the system of linear homogeneous equations,

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \qquad (i = 1, 2, \cdots, m), \tag{1}$$

where the a_{ij} are elements of a *commutative* ring R and solutions are sought in R. Let M denote the matrix of coefficients of the unknowns in this system of equations. The matrix M may be said to be of *rank* r if the determinant* of every square minor of M of order $r+1$ is annihilated by some one element of R, and the same is not true for minors of order r. M is of rank zero if all elements of M are annihilated by a fixed element of R. Note that if the ring R has no divisors of zero, these definitions agree with the familiar ones.

Obviously, the equations (1) always have the *trivial* solution $(0, 0, \cdots, 0)$. We shall now prove the following.

THEOREM 1. *The system* (1) *has a non-trivial solution if and only if the rank of the matrix of the coefficients is less than the number of unknowns.*

The proof is essentially that to be found in most texts on the theory of equations, with simple modifications to take care of the possible presence of divisors of zero in R. Suppose first that there exists a non-trivial solution $(x_1, x_2, \cdots, x_n)$ with $x_k \neq 0$. If $m < n$, clearly the rank r of M is less than n. If $m \geqq n$, consider a fixed determinant D of order n which can be found in M. For simplicity of statement suppose it comes from the first n rows of M. Multiply the first equation by the co-factor of a_{1k} in D, the second by the co-factor of a_{2k} in D, $\cdots$, the nth by the co-factor of a_{nk} in D, and add. There results, $x_kD = 0$. Similarly, it will be seen that x_k annihilates the determinant of every minor of M of order n, and thus the rank of M is less than n.

Now let us assume that the rank of M is $r < n$, and let $d \neq 0$ be an element of R which annihilates the determinants of all minors of M of order $r+1$. If $r = 0$, then clearly $x_j = d (j = 1, 2, \cdots, n)$ is a non-trivial solution of equations (1). If $r > 0$, then the product of d by the determinant of some minor of M of order r is not zero. Let us assume for convenience of notation that this minor is in the

* We shall not go into the question here, but it is not difficult to show that the usual theorems on expansions of a determinant are valid, if the elements lie in an arbitrary commutative ring. See, e.g., McCoy [6], p. 281.

upper left-hand corner of M. Let $c_1, c_2, \cdots, c_{r+1}$ be the co-factors of the elements in the last row of the determinant of order $r+1$ in the upper left-hand corner.* Then we assert that

$$x_j = dc_j \qquad (j = 1, 2, \cdots, r+1),$$
$$x_j = 0 \qquad (j = r+2, \cdots, n),$$

in a solution of equations (1), and is certainly non-trivial as $x_{r+1} \neq 0$. That these values satisfy the first r equations comes from the fact that the sum of the products of the elements of any row of a determinant by the co-factors of the corresponding elements of another row, is always zero. The remaining equations are also satisfied since when these values are substituted for the unknowns, we get the product of d by the determinant of a square minor of M of order $r+1$, which vanishes by hypothesis. The theorem is therefore established.

We shall obtain one further result before proceeding to a different topic. Let A be a given element of the matrix ring R_n, that is, A is a matrix with n rows and n columns and with elements in the commutative ring R. Let us seek an element X of R_n such that $AX = 0$. It is clear from the definition of multiplication of matrices that each column of X must be a solution of a system of linear homogeneous equations, the matrix of whose coefficients is precisely A. Thus, by the theorem just established, there exists a matrix X, other than the zero matrix, such that $AX = 0$ if, and only if, the determinant of A is a divisor of zero in R. A similar conclusion follows from a consideration of the equation $XA = 0$. We thus have the

COROLLARY.† *An element A of R_n is a divisor of zero in R_n if and only if the determinant of A is a divisor of zero in R.*

3. Polynomial rings. If R is a given ring, we may construct a new ring in the following almost obvious way. Let λ be a new symbol or *indeterminate* and consider the totality of polynomials,

$$f(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n, \tag{2}$$

with coefficients a_i in R. If we define addition and multiplication of polynomials by the usual formal rules, assuming that λ is commutative with elements of R, it will be seen that the set of all such polynomials is a ring, which may be denoted by the symbol $R[\lambda]$. The ring $R[\lambda]$ contains all elements of R, the polynomials consisting only of "constant" terms, and we say therefore that R is a *subring* of $R[\lambda]$. We may point out that two polynomials are considered equal if, and only if, the coefficients of the different powers of λ are the same in both polynomials.

* If $r = m < n$ we may add to our system of equations another equation in which all coefficients are zero.

† This result, as well as some others having to do with divisors of zero in rings of matrices, will be found in [7].

If the element $f(\lambda)$ of $R[\lambda]$ defined by (2) has $a_0 \neq 0$, we say that $f(\lambda)$ has *degree n*, and call a_0 the *leading coefficient* of $f(\lambda)$. It does not follow, however, that the degree of a product of two polynomials is always equal to the sum of the degrees of the factors. As an illustration of an extreme case, but one in which we are particularly interested, suppose for the moment that R is the ring I_{12} of integers modulo 12. Then clearly

$$(4\lambda^2 + 8\lambda + 4)(3\lambda + 6) = 0.$$

Thus $f(\lambda) = 4\lambda^2 + 8\lambda + 4$ is annihilated by a polynomial of the first degree, namely $3\lambda + 6$. But it is obvious that $f(\lambda)$ is in fact annihilated by an element of I_{12}, e.g. the element 3. This is an almost trivial illustration of the following general result.

THEOREM 2. *If R is a commutative ring and an element $f(\lambda)$ of $R[\lambda]$ is a divisor of zero in $R[\lambda]$, then $f(\lambda)$ is annihilated by an element of R.*

One method of proof of this theorem will be illustrated by a special case. Let

$$f(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \qquad a_0 \neq 0,$$

and

$$g(\lambda) = b_0\lambda^2 + b_1\lambda + b_2, \qquad b_0 \neq 0,$$

be elements of $R[\lambda]$ such that $f(\lambda)g(\lambda) = 0$. Multiplying, and collecting coefficients of the different powers of λ, we see therefore that

$$\begin{aligned} a_0b_0 &= 0, \\ a_1b_0 + a_0b_1 &= 0, \\ a_2b_0 + a_1b_1 + a_0b_2 &= 0, \\ a_3b_0 + a_2b_1 + a_1b_2 &= 0, \\ a_3b_1 + a_2b_2 &= 0, \\ a_3b_2 &= 0. \end{aligned} \tag{3}$$

This says, since $b_0 \neq 0$, that (b_0, b_1, b_2) is a non-trivial solution of a certain system of linear homogeneous equations. The first part of the proof of Theorem 1 then shows that b_0 annihilates the determinant of every minor of order three which can be chosen from the matrix of a's. In particular,

$$0 = b_0 \begin{vmatrix} a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \end{vmatrix} = b_0a_1^3, \tag{4}$$

since a_1^3 is the only term in the expansion of this determinant which does not contain a_0 as a factor, and $a_0b_0 = 0$ by the first equation of the system (3). Thus

there exists an integer j_1 with $0 \leqq j_1 \leqq 2$ such that $b_0 a_1^{j_1} \neq 0$ but $b_0 a_1^{j_1+1} = 0$.* Then $b_0 a_1^{j_1}$ annihilates both a_0 and a_1. Now applying the same argument as was used to get (4), we see that

$$0 = b_0 a_1^{j_1} \begin{vmatrix} a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \\ 0 & a_3 & a_2 \end{vmatrix} = b_0 a_1^{j_1} a_2^3. \tag{5}$$

Now choose an integer j_2 with $0 \leqq j_2 \leqq 2$ such that $b_0 a_1^{j_1} a_2^{j_2} \neq 0$ but $b_0 a_1^{j_1} a_2^{j_2+1} = 0$. Thus $b_0 a_1^{j_1} a_2^{j_2}$ annihilates a_0, a_1 and a_2. Then finally, by using the last three equations of (3), we see

$$0 = b_0 a_1^{j_1} a_2^{j_2} \begin{vmatrix} a_3 & a_2 & a_1 \\ 0 & a_3 & a_2 \\ 0 & 0 & a_3 \end{vmatrix} = b_0 a_1^{j_1} a_2^{j_2} a_3^3.$$

Again choose an integer j_3 such that $b_0 a_1^{j_1} a_2^{j_2} a_3^{j_3} \neq 0$, but $b_0 a_1^{j_1} a_2^{j_2} a_3^{j_3+1} = 0$. This element $c = b_0 a_1^{j_1} a_2^{j_2} a_3^{j_3}$ then is an element of R which annihilates all coefficients in $f(\lambda)$, as required by the theorem. A general proof can be given along lines of this calculation, the only difficulty being one of notation. An inductive proof of this theorem has also been obtained by Mrs. Alexandra Illmer Forsythe.

We note that the method of proof of this theorem outlined above shows that if b_0 is the leading coefficient of *any* annihilator of $f(\lambda)$, then $f(\lambda)$ is annihilated by cb_0 for proper choice of c in R. This observation will be used presently.

If now μ is another indeterminate, we may introduce the ring of polynomials in λ and μ, with coefficients from R, and this ring may be denoted by $R[\lambda, \mu]$. Elements of $R[\lambda, \mu]$ may also be considered as polynomials in μ with coefficients from $R[\lambda]$, or as polynomials in λ with coefficients from $R[\mu]$. It is now easy to prove

THEOREM 3. *If R is a commutative ring and an element $f(\lambda, \mu)$ of $R[\lambda, \mu]$ is a divisor of zero in $R[\lambda, \mu]$, then $f(\lambda, \mu)$ is annihilated by an element of R.*

Let

$$f(\lambda, \mu) = f_0(\mu)\lambda^n + f_1(\mu)\lambda^{n-1} + \cdots + f_n(\mu), \qquad f_0(\mu) \not\equiv 0,$$

where the $f_i(\mu)$ are elements of $R[\mu]$. Since $f(\lambda, \mu)$ is a divisor of zero in $R[\lambda, \mu]$, the preceding theorem, with R replaced by $R[\mu]$, states that there exists an element $h(\mu)$ of $R[\mu]$ which annihilates $f(\lambda, \mu)$, thus

$$f_i(\mu)h(\mu) = 0 \qquad (i = 0, 1, \cdots, n). \tag{6}$$

Suppose $h(\mu)$ has leading coefficient b_0, and consider the first of the equations (6). Then the observation made above shows that there is an element c_0 of R such that $c_0 b_0$ annihilates $f_0(\mu)$. Then clearly

* By a_1 we shall mean the unit element 1 of R.

$$f_1(\mu)[c_0h(\mu)] = 0, \qquad c_0b_0 \neq 0,$$

and again since c_0b_0 is the leading coefficient of an annihilator of $f_1(\mu)$, there is an element c_1 of R such that $c_1c_0b_0$ annihilates $f_1(\mu)$, and it clearly also annihilates $f_0(\mu)$. A repetition of this argument finally leads to the existence of elements $c_0, c_1, \cdots, c_n$ of R such that $c_0c_1 \cdots c_nb_0$ annihilates all $f_i(\mu)$ and therefore annihilates $f(\lambda, \mu)$. This argument can naturally be just as well applied to prove the corresponding theorem for the case of polynomials in more than two indeterminates.

4. Rings without divisors of zero. So far we have been considering rings with divisors of zero; we may now mention briefly rings without divisors of zero. First we need another definition. A ring R, not necessarily commutative, in which the equations $ax=b$ and $ya=b$ always have solutions for arbitrary b in R and arbitrary $a\neq 0$ in R, is called a *quasi-field*. A commutative quasi-field is called simply a *field*. The ring of rational numbers is a field, as is also the ring of real numbers or the ring of complex numbers. It can also be shown that the ring of integers modulo p, where p is a prime, is a field with exactly p elements. These are simple examples but there are many others, and a careful study of fields is one of the most important subjects of study of modern abstract algebra. For the moment we are, however, principally interested in the almost obvious observation that a quasi-field can have no divisors of zero. For if $cd=0$, $c\neq 0$, multiplication on the left by the solution x of the equation $xc=1$, shows that $d=0$. It is furthermore clear that no subring of a quasi-field can have divisors of zero or, in other words, no ring which can be imbedded in a quasi-field can have divisors of zero. The following converse of this statement, for the commutative case only, is well known.

THEOREM 4. *A commutative ring without divisors of zero is a subring of a field.*

We shall not give the proof of this theorem, as it is to be found in all texts on abstract algebra.* Suffice it to say that the proof is entirely analogous to the method by which the rationals may be logically obtained from the integers by introduction of *formal quotients*, *i.e.*, pairs of integers (a, b), $b\neq 0$, having assigned properties of the familiar quotients a/b.

When we pass to non-commutative rings, the situation is much more complicated. Malcev [3] has given an example of a ring without divisors of zero which is not a subring of any quasi-field, so that we cannot hope to establish Theorem 4 for the non-commutative case. However, Ore [8] has made some progress in this direction by a careful study of formal quotients. The main result of Ore in this connection is the following

THEOREM 5. *A ring R, without divisors of zero, can be imbedded in a quasi-field by use of formal quotients if, and only if, for every pair a, b of non-zero elements of R, there exist non-zero elements m, n of R such that*

* See, *e.g.*, van der Waerden [11], p. 47.

(7) $$am = bn.$$

It will be noticed that, if R is commutative, we may choose $m=b$, $n=a$, so that Theorem 5 is actually a direct generalization of Theorem 4. Theorem 5 states what can be accomplished by use of formal quotients, but it does not exclude the possibility of some other imbedding process. Thus, whether condition (7) is a necessary property of every subring of an arbitrary quasi-field is an interesting and unsolved problem.

5. Rings without nilpotent elements. We henceforth restrict our attention to commutative rings. It may happen, as an extreme case, that a non-zero element a of a ring R has the property that for a suitably chosen positive integer n, $a^n=0$, in which case a is said to be *nilpotent*. Thus, for example, in the ring I_4, we have $2^2=0$. Clearly a nilpotent element is a special kind of a divisor of zero. Now it happens that rings without nilpotent elements are, on the whole, much better behaved than rings with nilpotent elements. We shall presently state a theorem giving a certain characterization of commutative rings without nilpotent elements. It should be emphasized, however, that a ring without nilpotent elements may have divisors of zero. Thus in the ring I_6 there are no nilpotent elements, but $2\cdot 3=0$.

We now need a further definition, Let R_1 and R_2 be two rings, not necessarily distinct, and consider the set of all pairs (a_1, a_2) where a_1 is an element of R_1 and a_2 an element of R_2. Define addition and multiplication of pairs as follows:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2),$$

$$(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2),$$

it being naturally understood that a_1+b_1 is addition in R_1, a_2+b_2 is addition in R_2, and so on. With these definitions, it is found that the set of all such pairs is a ring which is called the *direct sum* of the rings R_1 and R_2, and written as $R_1\dotplus R_2$. In this direct sum, the zero element is the pair (0, 0) and the unit element is (1, 1). Clearly $R_1\dotplus R_2$ will have divisors of zero even if R_1 and R_2 have none, for example, we have $(1, 0)\ (0, 1)=(0, 0)$. But, for our purposes, it is significant and easily verified that if R_1 and R_2 have no nilpotent elements neither does their direct sum. We may pause to illustrate these concepts, as well as some other points, by an example. The ring $I_2\dotplus I_3$ consists of the six elements (0, 0), (0, 1), (0, 2), (1, 0), (1, 1) and (1, 2). Let us set up a one-to-one correspondence between these six elements and the elements $0, 1, 2, 3, 4, 5$ of the ring I_6 as follows:

$$0 \leftrightarrow (0, 0) \qquad 3 \leftrightarrow (1, 0)$$

$$1 \leftrightarrow (1, 1) \qquad 4 \leftrightarrow (0, 1)$$

$$2 \leftrightarrow (0, 1) \qquad 5 \leftrightarrow (1, 2)$$

It may now be observed that if, in this correspondence, $a\leftrightarrow(b, c)$ and $d\leftrightarrow(e, f)$, then $a+d\leftrightarrow(b, c)+(e, f)$ and $ad\leftrightarrow(b, c)\ (e, f)$. Thus the ring I_6 is the same ring

as $I_2 \dot{+} I_3$ except for the notation employed. It is customary to say that these rings are therefore *isomorphic*. Since the ring of integers modulo a prime is actually a field, we have shown that the ring I_6 is isomorphic to a direct sum of two fields.* It is this kind of result which we now propose to generalize. But first we need to extend our definition of direct sum.

In the definition given above, we limited ourselves to the direct sum of two rings, but it is almost obvious that a corresponding definition can be given for the direct sum of any finite, or even infinite, set of rings. In order to illustrate the general notation to be used presently, let us re-formulate the definition of $R_1 \dot{+} R_2$ in a different, but equivalent, way. Let $\mathfrak{N}$ be the set consisting of two elements α_1 and α_2. Then $R_1 \dot{+} R_2$ may be considered as the set of all functions $f(x)$ defined on $\mathfrak{N}$, with $f(\alpha_i)$ having values in $R_i (i = 1, 2)$, sums and products of functions being defined in the usual way. It is now easy to formulate a general definition of direct sum. Let $\mathfrak{M}$ be an arbitrary set such that to each element α of $\mathfrak{M}$ there corresponds a ring R_α. The set of all functions $f(x)$, defined on $\mathfrak{M}$, such that for every element α of $\mathfrak{M}$, $f(\alpha)$ is in R_α, is a ring which we define to be the *direct sum of the rings* R_α (α *in* $\mathfrak{M}$). It is not assumed that the rings R_α are all distinct. In fact, as a special case of some importance, they may all be identical.

We now state without proof our final theorem.

THEOREM 6. *A commutative ring without nilpotent elements is isomorphic to a subring of a direct sum of fields.*

This theorem was explicitly stated in [5], it being an almost immediate consequence of a theorem of Krull [2] and a principle introduced by Montgomery and the author in [4].

We may remark that, in general, it is not necessarily true that a commutative ring without nilpotent elements is isomorphic to a full direct sum of fields, but only to a subring of such a direct sum. However, it may happen as a special case that the subring consists of the entire ring. This was true in the example of I_6, as we showed it to be isomorphic to $I_2 \dot{+} I_3$.

We conclude with an application of Theorem 6 of some interest. Stone [10] has defined a *Boolean ring* as a ring B such that $a^2 = a$ for every element a of B. It is easy to show that B is necessarily commutative and also that $a + a = 0$ for every a in B. Henceforth, let B denote a fixed Boolean ring. Since $a^2 = a$, no element can be nilpotent and thus, by the preceding theorem, there exists a set $\mathfrak{M}$ and fields $F_\alpha (\alpha \text{ in } \mathfrak{M})$ such that B is isomorphic to a subring of the direct sum of the fields $F_\alpha (\alpha \text{ in } \mathfrak{M})$. Hence to each element a of B we may make correspond a unique element $f_a(x)$ of this direct sum, and the ring of functions $f_a(x)$ (a in B) is isomorphic to B. Now it can be shown, although we shall omit the proof, that in the case of Boolean rings each of the fields F_α may be taken to be the field

* We may remark that $R_1 \dot{+} R_2$ and $R_2 \dot{+} R_1$ are clearly isomorphic. Hence in speaking of the direct sum of rings we do not need to specify any definite order in which the rings occur.

I_2 of integers modulo 2; thus it has only two elements 0 and 1. To each element a of B let us now associate the subset $\mathfrak{M}_a$ of $\mathfrak{M}$ consisting of all points α of $\mathfrak{M}$ such that $f_a(\alpha)=1$. If $a\neq b$, clearly $f_a(x)\neq f_b(x)$ and therefore $\mathfrak{M}_a\neq\mathfrak{M}_b$. This correspondence, which we may indicate by $a\leftrightarrow\mathfrak{M}_a$ is clearly a one-to-one correspondence between the elements of B and a certain class of subsets of $\mathfrak{M}$. Since now in I_2, $1+1=0$, it is easy to verify that if $a\leftrightarrow\mathfrak{M}_a$ and $b\leftrightarrow\mathfrak{M}_b$, then ab corresponds to the intersection of $\mathfrak{M}_a$ and $\mathfrak{M}_b$ while $a+b$ corresponds to the set of points of $\mathfrak{M}$ which are in $\mathfrak{M}_a$ or in $\mathfrak{M}_b$ but not in both. A correspondence of this type has been called by Stone a *representation* of the Boolean ring B. We have therefore sketched a proof that every Boolean ring has a representation. A number of different proofs of this fact are to be found in the literature.* For applications of the representation theory, as well as the relation of Boolean rings to the Boolean algebras of logic, reference may be made to a series of papers by Stone, particularly [9] and [10] listed below.

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* See Stone [9], Frink [1] and the further references given in this latter paper.

THE CESÀRO KERNEL TRANSFORMATION

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1. Introduction. In this paper we are concerned with transformations of the type

$$z(s) = \int_0^s k(s, t)x(t)dt, \tag{1.1}$$

where $x(s)$ is bounded and integrable, $0 \leqq s \leqq s_1$, and the *kernel* $k(s, t)$ is integrable in t for each t, $0 \leqq t \leqq s$. The transformation or the kernel is said to be *regular* if $\lim_{s\to\infty} x(s)$ implies the existence of $\lim_{s\to\infty} z(s)$ and the equality of the two limits [1]. In particular, the kernel

$$k(s, t) = \alpha(1 - t/s)^{\alpha-1}/s, \qquad \alpha > 0,$$

defines Cesàro summability (C, α), $\alpha > 0$.

To illustrate the use of regular kernel transformations in assigning values to functions $x(t)$, where $\lim_{t\to\infty} x(t)$ fails to exist, we show that summability $(C, 1)$ assigns the value 0 to the function $x(t) = \sin t$. Indeed, for the example at hand, we have simply

$$z(s) = \frac{1}{s}\int_0^s \sin t\, dt = \frac{1}{s}(1 - \cos s) \to 0 \quad \text{as} \quad s \to \infty.$$

Various sets of regularity conditions for the transformation (1.1) have been given by Agnew [2], Knopp [3], and Silverman [1]. We reproduce here a set of regularity conditions written by Silverman which we find convenient to use in this paper.

THEOREM 1. *Let $k(s, t)$ be defined, $0 < s$, $0 < t < s$, and integrable in t for each s; then sufficient conditions that $k(s, t)$ shall be a regular kernel are*

$$\lim_{s\to\infty} \int_0^s k(s, t)dt = 1, \tag{i}$$

$$\lim_{s\to\infty} k(s, t) = 0 \ \textit{uniformly in } t, \ 0 \leqq t \leqq q, \tag{ii}$$

$$\int_0^s k(s, t)dt < A, \qquad 0 < s, \tag{iii}$$

where q is an arbitrary constant and A is a positive constant.

Let us designate the method of summation defined by (1.1) by the symbol (k). It is the object of this paper to determine conditions on a regular kernel $k(s, t)$ in order that $(k) \supset (C, \alpha)$, $\alpha > 0$. It will be evident from the discussion which follows that two cases have to be considered according as α is an integer or is not an integer. Two theorems are obtained and examples given in illustration of them.

2. The non-integral case. The main result of this section is contained in the following theorem.

THEOREM 2. $(k)\supset(C,\alpha)$, $\alpha=n+\beta$, $0<\beta<1$, $(n=0, 1, 2, \cdots)$, *provided that the kernel*

$$K(s,t)=\frac{(-1)^{n+1}t^{n+\beta}}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\frac{\partial^{n+1}}{\partial t^{n+1}}\omega(s,t), \tag{2.1}$$

where

$$\omega(s,t)=\int_t^s\frac{k(s,u)}{(u-t)^{\beta}}du,$$

is regular, and provided that

$$\frac{(-1)^{n+1}}{\Gamma(1-\beta)\Gamma(n+\beta)}\int_t^s(u-t)^{n+\beta-1}\frac{\partial^{n+1}}{\partial u^{n+1}}\omega(s,u)du=k(s,t) \tag{2.2}$$

is an identity.

The Cesàro transformation for summability (C,α), $\alpha>0$, is

$$y(t)=\frac{\alpha}{t}\int_0^t\left(1-\frac{u}{t}\right)^{\alpha-1}x(u)du. \tag{2.3}$$

We find it convenient to set $\alpha=n+\beta$, $0<\beta\leqq 1$, $(n=0, 1, 2, \cdots)$. There are two cases to be considered according as $\beta=1$ or not. The case $\beta=1$ is the simplest case and will be considered in section 4.

For the case $\beta\neq 1$ we wish to determine conditions on the regular kernel $k(s,t)$ so that $\lim_{s\to\infty}y(s)=l$ implies $\lim_{s\to\infty}z(s)=l$. In this connection the problem of expressing z in terms of y presents itself. We do this formally at first, making all necessary assumptions to determine the *form* of the required transformation. Our formal method of procedure will be to solve (2.3) for x in terms of y and then substitute in (1.1), obtaining z in terms of y.

To solve (2.3) for x in terms of y we begin by writing the equation (2.3) in the form

$$t^{n+\beta}y(t)=(n+\beta)\int_0^t(t-u)^{n+\beta-1}x(u)du, \tag{2.4}$$

and setting

$$v_1(u)=\int_0^u x(w)dw,\ v_2(u)=\int_0^u v_1(w)dw,\ \cdots,\ v_n(u)=\int_0^u v_{n-1}(w)dw.$$

Integrating by parts in (2.4) we get

$$t^{n+\beta}y(t)=(n+\beta)(n+\beta-1)\int_0^t(t-u)^{n+\beta-2}v_1(u)du.$$

After n successive integrations by parts we obtain

$$\frac{\Gamma(\beta)t^{n+\beta}y(t)}{\Gamma(n+\beta+1)} = \int_0^t (t-u)^{\beta-1}v_n(u)du.$$

This equation is now in the form of an Abel's integral equation which we can solve for v_n in terms of y [4] to obtain

$$v_n(t) = \frac{1}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\frac{d}{dt}\int_0^t \frac{u^{n+\beta}y(u)}{(t-u)^{\beta}}du.$$

Differentiating this equation n times with respect to t we get

$$x(t) = \frac{1}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\frac{d^{n+1}}{dt^{n+1}}\int_0^t \frac{u^{n+\beta}y(u)}{(t-u)^{\beta}}du. \tag{2.5}$$

Now, eliminating x between (1.1) and (2.5), we have

$$z(s) = \frac{1}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\int_0^s k(s,t)\frac{d^{n+1}}{dt^{n+1}}\int_0^t \frac{u^{n+\beta}y(u)}{(t-u)^{\beta}}du\,dt,$$

or

$$z(s) = \frac{1}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\int_0^s k(s,t)\int_0^t \frac{D_u^{n+1}\{u^{n+\beta}y(u)\}}{(t-u)^{\beta}}du\,dt.$$

In order to find the kernel of this transformation we first interchange the order of integration to obtain

$$z(s) = \frac{1}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\int_0^s D_u^{n+1}\{u^{n+\beta}y(u)\}\omega(s,u)du,$$

with $\omega(s, u)$ as previously defined. Integrating by parts $n+1$ times we have finally

$$z(s) = \frac{(-1)^{n+1}}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\int_0^s \frac{\partial^{n+1}}{\partial u^{n+1}}\omega(s,u)\cdot u^{n+\beta}y(u)du. \tag{2.6}$$

It is clear that this is not a valid transformation from y to z unless we impose restrictions of unwarranted severity on the class of functions x which we are considering. Accordingly, we shall look for a means of obtaining (2.6) directly without further limiting the class of bounded and integrable functions x. To this end we eliminate y between (2.3) and (2.6) and attempt to retrace our way to (1.1) without adding to the original restrictions on x. Thus, we have

$$\begin{aligned} z(s) &= \frac{(-1)^{n+1}}{\Gamma(1-\beta)\Gamma(n+\beta)}\int_0^s \frac{\partial^{n+1}}{\partial u^{n+1}}\omega(s,u)\int_0^u (u-t)^{n+\beta-1}x(t)dt\,du \\ &= \frac{(-1)^{n+1}}{\Gamma(1-\beta)\Gamma(n+\beta)}\int_0^s x(t)\int_t^s (u-t)^{n+\beta-1}\frac{\partial^{n+1}}{\partial u^{n+1}}\omega(s,u)du\,dt. \end{aligned}$$

If this transformation is to reduce to (1.1) we must have

$$(2.7)\qquad \frac{(-1)^{n+1}}{\Gamma(1-\beta)\Gamma(n+\beta)}\int_t^s (u-t)^{n+\beta-1}\frac{\partial^{n+1}}{\partial u^{n+1}}\omega(s,u)du = k(s,t).$$

Accordingly, to obtain (2.6) directly we must establish (2.7) as an identity.

Finally, the kernel $K(s,t)$ of (2.6) must be regular in order that $\lim_{s\to\infty} y(s)=l$ shall imply $\lim_{s\to\infty} z(s)=l$.

3. An illustration. To illustrate the use of Theorem 2 we shall prove the known result, $(C,\gamma)\supset(C,n+\beta)$, $\gamma>n+\beta$, $0<\beta<1$, $(n=0,1,2,\cdots)$.

To prove the identity (2.2) we replace $k(s,t)$ in the left member by the kernel for summability (C,γ) and write

$$\begin{aligned}
&\frac{\gamma(-1)^{n+1}}{s^\gamma\Gamma(1-\beta)\Gamma(n+\beta)}\int_t^s (u-t)^{n+\beta-1}\frac{\partial^{n+1}}{\partial u^{n+1}}\int_u^s (s-t)^{\gamma-1}(t-u)^{-\beta}dt\,du\\
&\quad=\frac{\Gamma(\gamma+1)(-1)^{n+1}}{s^\gamma\Gamma(\gamma+1-\beta)\Gamma(n+\beta)}\int_t^s (u-t)^{n+\beta-1}\frac{\partial^{n+1}}{\partial u^{n+1}}(s-u)^{\gamma-\beta}du\\
&\quad=\frac{\Gamma(\gamma+1)}{s^\gamma\Gamma(\gamma-\beta-n)\Gamma(n+\beta)}\int_t^s (u-t)^{n+\beta-1}(s-u)^{\gamma-\beta-n-1}du\\
&\quad=\frac{\gamma}{s^\gamma}(s-t)^{\gamma-1}.
\end{aligned}$$

It remains to prove that the kernel $K(s,t)$ of the transformation (2.6), with $k(s,t)$ replaced by the kernel for summability (C,γ), is regular. We have

$$\begin{aligned}
K(s,t) &= \frac{\gamma(-1)^{n+1}t^{n+\beta}}{s^\gamma\Gamma(1-\beta)\Gamma(n+\beta+1)}\frac{\partial^{n+1}}{\partial t^{n+1}}\int_t^s (s-u)^{\gamma-1}(u-t)^{-\beta}du\\
&= \frac{(-1)^{n+1}\Gamma(\gamma+1)t^{n+\beta}}{s^\gamma\Gamma(n+\beta+1)\Gamma(\gamma+1-\beta)}\frac{\partial^{n+1}}{\partial t^{n+1}}(s-t)^{\gamma-\beta}\\
&= \frac{\Gamma(\gamma+1)t^{n+\beta}(s-t)^{\gamma-\beta-n-1}}{\Gamma(\gamma-\beta-n)\Gamma(n+\beta+1)s^\gamma}\\
&= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta-n)\Gamma(n+\beta+1)}\frac{t^{n+\beta}}{s^{n+\beta+1}}\left(1-\frac{t}{s}\right)^{\gamma-\beta-n-1}.
\end{aligned}$$

It is clear that $\lim_{s\to\infty} K(s,t)=0$ uniformly in t, $0\leqq t\leqq q$, where q is an arbitrary constant. Accordingly, condition (ii) of Theorem 1 is satisfied. Moreover, since

$$\int_0^s K(s,t)dt = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta-n)\Gamma(n+\beta+1)s^\gamma}\int_0^s t^{n+\beta}(s-t)^{\gamma-\beta-n-1}dt = 1,$$

conditions (i) and (iii) of Theorem 1 are also fulfilled. This completes the proof of this section.

4. The integral case. The theorem with which this section is occupied is much easier to establish and to apply than Theorem 2.

THEOREM 3. $(k)\supset(C, n)$, $(n=1, 2, 3, \cdots)$, *provided that the kernel*

$$K(s, t) = \frac{(-1)^n}{\Gamma(n+1)} t^n \frac{\partial^n}{\partial t^n} k(s, t) \tag{4.1}$$

is regular, and provided that

$$\frac{\partial^i}{\partial t^i} k(s, t)\Big]_{t=s} = 0, \qquad i = 0, 1, 2, \cdots, n-1. \tag{4.2}$$

To prove this theorem we follow almost exactly the procedure of section 2. First of all we set $\alpha=n$ in (2.4) to get

$$t^n y(t) = n \int_0^t (t-u)^{n-1} x(u)du. \tag{4.3}$$

Using the notation of section 2 we obtain, after $n-1$ integrations by parts,

$$t^n y(t) = \Gamma(n+1) \int_0^t v_{n-1}(u)du.$$

After n successive differentiations with respect to t we get

$$x(t) = \frac{1}{\Gamma(n+1)} D_t^n \{t^n y(t)\}. \tag{4.4}$$

Now, eliminating x between (1.1) and (4.4), we have

$$z(s) = \frac{1}{\Gamma(n+1)} \int_0^s k(s, t) D_t^n \{t^n y(t)\} dt.$$

In order to find the kernel of this transformation we first integrate by parts n times; then with the aid of (4.2) we have

$$z(s) = \frac{(-1)^n}{\Gamma(n+1)} \int_0^s t^n \frac{\partial^n}{\partial t^n} k(s, t) y(t)dt. \tag{4.5}$$

Thus, once again, we obtain formally the transformation from y to z. As in section 2 we now look for a direct means of obtaining (4.5). To this end we eliminate y between (4.3) and (4.5) and attempt to retrace our way to (1.1). We have

$$z(s) = \frac{(-1)^n}{\Gamma(n)} \int_0^s \frac{\partial^n}{\partial t^n} k(s, t) \int_0^t (t-u)^{n-1} x(u)du\, dt.$$

Interchanging the order of integration we obtain

$$z(s) = \frac{(-1)^n}{\Gamma(n)} \int_0^s x(u) \int_u^s (t-u)^{n-1} \frac{\partial^n}{\partial t^n} k(s, t)dt\, du.$$

Now, it remains to show that the kernel of this transformation:

$$\frac{(-1)^n}{\Gamma(n)} \int_u^s (t-u)^{n-1} \frac{\partial^n}{\partial t^n} k(s, t)dt, \tag{4.6}$$

reduces to $k(s, u)$. Using (4.2) and integrating by parts $n-1$ times in (4.6) we get

$$-\int_u^s \frac{\partial}{\partial t} k(s, t)dt = k(s, u).$$

Since the steps in these operations are reversible we are thus able to obtain the transformation (4.5) directly without restricting the class of bounded and integrable functions x which we are studying. This completes the proof of Theorem 3.

5. An illustration. To illustrate the use of Theorem 3 we shall check the known result $(C, \gamma) \supset (C, n)$, $\gamma > n$. We observe first of all that the kernel associated with summability (C, γ) satisfies the requirement (4.2) for $\gamma > n$. It remains to prove that the kernel $K(s, t)$, as defined by (4.1), with $k(s, t)$ replaced by the kernel for summability (C, γ), $\gamma > n$, is regular. We set $\gamma = n + \delta$, $0 < \delta < 1$, and write

$$\begin{aligned} K(s, t) &= \frac{(-1)^n(n+\delta)}{\Gamma(n+1)} \frac{t^n}{s^{n+\delta}} \frac{\partial^n}{\partial t^n} (s-t)^{n+\delta-1} \\ &= \frac{\Gamma(n+\delta+1)}{\Gamma(n+1)\Gamma(\delta)} \frac{t^n}{s^{n+\delta}} (s-t)^{\delta-1} \\ &= \frac{\Gamma(n+\delta+1)}{\Gamma(n+1)\Gamma(\delta)} \frac{t^n}{s^{n+1}} \left(1 - \frac{t}{s}\right)^{\delta-1}. \end{aligned}$$

We see by inspection that condition (ii) of the regularity conditions in Theorem 1 is fulfilled. Moreover, we have at once

$$\int_0^s K(s, t)dt = \frac{\Gamma(n+\delta+1)}{\Gamma(n+1)\Gamma(\delta)s^{n-\delta}} \int_0^s t^n(s-t)^{\delta-1}dt = 1.$$

Thus, conditions (i) and (iii) of Theorem 1 are also satisfied.

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THE EULER-DIDEROT ANECDOTE

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No anecdote with regard to a mathematician is better known than the story of the discomfiture of Diderot by Euler. The story was first told by Thiébault [1]; it was later retold, with additions, by De Morgan [2]. Since then a great many authors, all following the more highly colored version of De Morgan, have repeated the story. It is the purpose of this note to show that one addition by De Morgan—the addition which really gives point to the story—is manifestly absurd, and that the credibility of the original story by Thiébault is open to suspicion.

It will be sufficient to give the De Morgan version, noting what he added:

"The following story is told by Thiébault, in his *Souvenirs de vingt ans de séjour à Berlin*, published in his old age, about 1804. This volume was fully received as trustworthy; and Marshall Mollendorff told the Duc de Bassano in 1807 that it was the most veracious of books written by the most honest of men. Thiébault says that he has no personal knowledge of the truth of the story, but that it was believed throughout the whole of the north of Europe. Diderot paid a visit to the Russian court at the invitation of the Empress. He conversed very freely, and gave the younger members of the court circle a good deal of lively atheism. The Empress was much amused, but some of the councillors suggested that it might be desirable to check these expositions of doctrine. The Empress did not like to put a direct muzzle on her guest's tongue, so the following plot was contrived. Diderot was informed that a learned mathematician was in possession of an algebraical demonstration of the existence of God, and would give it to him before all the court, if he desired to hear it. Diderot gladly consented; though the name of the mathematician is not given, it was Euler. He advanced toward Diderot, and said gravely and in a tone of perfect conviction: *Monsieur*, $(a+b^n)/n=x$, *donc Dieu existe*; *répondez*! Diderot, to whom algebra was Hebrew, was embarrassed and disconcerted; while peals of laughter rose on all sides. He asked permission to return to France at once, which was granted."

This differs from the Thiébault account in three respects:

(1) The formula is slightly different; this affects neither the validity of the proof nor the credibility of the story.

(2) It identifies the mathematician as Euler.

(3) The expression "to whom algebra was Hebrew" is an addition. Thiébault says: "Diderot, voulant prouver la nullité et l'ineptie de cette prétendue preuve, mais ressentant malgré lui, l'embarras où l'on est d'abord lorsqu'on découvre chez les autres, le dessein de nous jour, n'avoit pu échapper aux plaisanteries dont on étoit prêt à l'assaillir."

Since then the story has, as I have said, been repeated many times. To cite only two instances—both of them by authors of popular books—we find Hogben

beginning his *Mathematics for the Million* with this story, but with the substitution of "Arabic" for "Hebrew," and Bell in his *Men of Mathematics* giving a modified version, but with "Chinese" for "Hebrew."

That is the story, and it is a very good story, except that it isn't true. To Diderot algebra was neither Hebrew, nor Arabic, nor even Chinese. Diderot was a very good mathematician, and prior to his Russian trip, was the author of five creditable memoirs on mathematics [3]. To mention only one of these, in the second memoir, *Examen de la développante du cercle*, Diderot shows that if, instead of the Euclidean tools of ruler and compass, we assume a circle and its involute, which last is easily constructed mechanically, then the classical problems, trisection of the angle, duplication of the cube, and quadrature of the circle, may be easily and neatly solved. In proving this, Diderot shows a complete mastery of algebra, geometry, and the calculus.

The anecdote as told by De Morgan and by all who have followed him, is thus seen to be absurd. But it may be noted that Thiébault's story is not so unskillful as to aver that Diderot could not reply; it merely says that he sensed the hostility of the audience. The Thiébault story *may* have been true; and the mathematician *may* have been Euler, who was in Russia at that time. What evidence is there for the original story? Thiébault, writing many years later, says merely that the story was believed throughout the north of Europe. No one else tells the story, in particular there is no known Russian source for the story. On the other hand it is known that Frederick the Great, King of Prussia, was a bitter enemy of Diderot. Several sources indicating that stories about Diderot at Saint Petersburg emanated from Berlin are cited and summarized by Tourneux [4].

The alternatives seem to be, first, a rather pointless incident as told by Thiébault; second, and more probable, a canard, inspired by Frederick the Great or by his courtiers.

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4. Diderot et Catherine II; Paris, 1899. I am indebted to Dr. Arthur M. Wilson, Professor of Biography at Dartmouth College for this last reference, and for his advance of the second alternative mentioned above.

JACOBIAN CIRCLES OF THE BIQUADRATIC

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1. Introduction. The transformations of inversive geometry consist of all homographies $w=(az+b)/(cz+d)$ and all antigraphies $\bar{w}=(az+b)/(cz+d)$, where, in each, $ad-bc\neq 0$, the letters represent complex numbers, and $\bar{w}$ is the conjugate of w. A homography of period 2 is a *polarity*; and an antigraphy of period 2 is an *inversion* [1].

Any inversion may be put in the form

$$(1.1) \qquad (A): \quad a_{11}z\bar{w} + a_{10}z + a_{01}\bar{w} + a_{00} = 0,$$

where $a_{ik}=\bar{a}_{ki}$, *i.e.*, a_{ii} is real. In the Argand plane an inversion sets up a (1, 1) correspondence between points z and w of the plane; and two corresponding points constitute a couple $[z, w]$ which represents a complex point on the locus defined by (1.1). We shall say that *the couple lies on the locus*. Since the equation (1.1) is self-conjugate, *i.e.*, real, the conjugate couple $[w, z]$ is also on the locus. The locus of a bilinear equation in z and $\bar{w}$ is called a circle, and when the equation is self-conjugate, as above, the locus is a real circle with or without a real trace [1]. If the circle (a) has a real trace, its real points z are represented by self-conjugate couples $[z, z]$ which satisfy the equation

$$a_{11}z\bar{z} + a_{10}z + a_{01}\bar{z} + a_{00} = 0.$$

This equation is also the locus of fixed points, if any, of the inversion (1.1).

The circle (a) is a *null* circle if and only if $a_{10}a_{01}-a_{11}a_{00}=0$; and the condition that two circles (a) and (a') be orthogonal [2] is

$$(1.2) \qquad a_{11}a'_{00} + a_{00}a'_{11} - a_{10}a'_{01} - a_{01}a'_{10} = 0.$$

A *biquadratic* is a curve which determines four couples with any circle. If any one of the couples is self-conjugate, it represents a real point of intersection of the two curves. The general equation of the biquadratic, therefore, is of the second degree in z and $\bar{w}$, and we write it

$$a_{22}z^2\bar{w}^2 + 2a_{21}z^2\bar{w} + 2a_{12}z\bar{w}^2 + a_{20}z^2 + 4a_{11}z\bar{w} + a_{02}\bar{w}^2 + 2a_{10}z + 2a_{01}\bar{w} + a_{00} = 0,$$

or, in the so-called matrix form,

$$(1.3) \qquad (B): \quad \begin{array}{c|ccc} & 1 & 2z & z^2 \\ \hline 1 & a_{00} & a_{10} & a_{20} \\ 2\bar{w} & a_{01} & a_{11} & a_{21} \\ \bar{w}^2 & a_{02} & a_{12} & a_{22} \end{array} = 0.$$

This equation is real if $a_{ik}=\bar{a}_{ki}$, *i.e.*, a_{ii} is real; and then the determinant $|a_{ik}|$ is Hermitian and is an inversive invariant of (B). We assume in the sequel that the equation of (B) is real.

Interpreted as a (2, 2) correspondence between points z and w of the Argand plane, equation (1.3) associates two image points w with every point z, and reciprocally.

2. Bipolar theory. If ξ is a given point and x its harmonic conjugate with respect to the images of w, as given by equation (1.3), we have a (1, 2) correspondence between points x and w of the plane, viz.,

$$(2.1) \qquad \begin{array}{c|ccc} & 1 & \xi + x & \xi x \\ \hline 1 & a_{00} & a_{10} & a_{20} \\ 2\bar{w} & a_{01} & a_{11} & a_{21} \\ \bar{w}^2 & a_{02} & a_{12} & a_{22} \end{array} = 0,$$

which associates one image point x with every point w; and with every point x, two images w. Consider now the harmonic conjugate y of a second given point η (which may coincide with ξ) with respect to the two images of x as given by equation (2.1). This sets up a (1, 1) correspondence between the points x and y of the plane, viz.,

$$(2.2) \qquad (\sigma): \quad \sigma_{11}x\bar{y} + \sigma_{10}x + \sigma_{01}\bar{y} + \sigma_{00} = 0,$$

where,

$$\begin{aligned} \sigma_{11} &= a_{11} + a_{12}\bar{\eta} + a_{21}\xi + a_{22}\xi\bar{\eta} \\ \sigma_{10} &= a_{10} + a_{11}\bar{\eta} + a_{20}\xi + a_{21}\xi\bar{\eta} \\ \sigma_{01} &= a_{01} + a_{02}\bar{\eta} + a_{11}\xi + a_{12}\xi\bar{\eta} \\ \sigma_{00} &= a_{00} + a_{01}\bar{\eta} + a_{10}\xi + a_{11}\xi\bar{\eta}. \end{aligned}$$

The equation (2.2) of the antigraphy between points x and y is also the equation of a complex circle associated with the couple $[\xi, \eta]$, and this circle is real when the couple is self-conjugate, *i.e.*, $\eta = \xi$.

DEFINITION 1. The *bipolar circle* (σ) of a couple $[\xi, \eta]$, with respect to the biquadratic, is the locus defined by equation (2.2).

The bipolar circle (σ) of a given couple $[\xi, \eta]$ is a linear combination of four circles, viz.,

$$(2.3) \qquad s_{11}\xi\bar{\eta} + s_{10}\xi + s_{01}\bar{\eta} + s_{00} = 0,$$

where the s_{ik} are the same functions of x and $\bar{y}$ that the σ_{ik} are of ξ and $\bar{\eta}$. Conversely, since any circle (c) may be written as a linear combination of these same four circles $s_{ik} = 0$, i.e.,

$$(c): \quad s_{11}k_{11} + s_{10}k_{10} + s_{01}k_{01} + s_{00}k_{00} = 0,$$

provided that

$$\Delta_4 = \begin{vmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ a_{10} & a_{11} & a_{20} & a_{21} \\ a_{01} & a_{02} & a_{11} & a_{12} \\ a_{00} & a_{01} & a_{10} & a_{11} \end{vmatrix} \neq 0,$$

such a circle (c) will be the bipolar circle of a couple $[\xi, \eta]$ if and only if

$$k_{11}k_{00} - k_{10}k_{01} = 0,$$

and then

$$\xi = k_{10}/k_{00} \quad \text{and} \quad \eta = \bar{k}_{01}/\bar{k}_{00}.$$

Consequently, every couple has a bipolar circle but, reciprocally, every circle is *not* the bipolar circle of a couple.

If $[x, y]$ is a couple on (σ), the bipolar of $[\xi, \eta]$, then the bipolar of $[x, y]$ may be written

$$s_{11}z\bar{w} + s_{10}z + s_{01}\bar{w} + s_{00} = 0,$$

which, by virtue of equation (2.3) is satisfied by $[\xi, \eta]$. This proves,

THEOREM 1. *The bipolar circles of all couples on the bipolar circle of a couple* $[\xi, \eta]$ *pass through (i.e., are orthogonal to)* $[\xi, \eta]$.

THEOREM 2. *The locus of couples whose bipolar circles are orthogonal to a given circle is a circle.*

This theorem follows from Definition 1 and equation (1.2). If the given circle is

$$(\alpha): \quad \alpha_{11}z\bar{w} + \alpha_{10}z + \alpha_{01}\bar{w} + \alpha_{00} = 0,$$

the locus of couples whose bipolars are orthogonal to (α) is the circle (β), where

$$\beta_{ik} = a_{ik}\alpha_{11} - a_{i,k+1}\alpha_{10} - a_{i+1,k}\alpha_{01} + a_{i+1,k+1}\alpha_{00}, \qquad (i, k = 0, 1). \tag{2.4}$$

DEFINITION 2. The *bipolar circle* (β) of a circle (α), with respect to the biquadratic, is the locus of couples whose bipolars are orthogonal to (α).

We observe that this definition includes Definition 1, for when (α) is the null circle $[\xi, \eta]$, *i.e.*, (α) has the equation

$$(z - \xi)(\bar{w} - \bar{\eta}) = 0,$$

then

$$\alpha_{11} = 1, \qquad \alpha_{10} = -\bar{\eta}, \qquad \alpha_{01} = -\xi, \qquad \alpha_{00} = \xi\bar{\eta},$$

and $\beta_{ik} = \sigma_{ik}$. Hence, equations (2.4) constitute a (1, 1) transformation of circles (including null circles) in the plane of the biquadratic.

THEOREM 3. *The bipolars of couples on a circle are orthogonal to a circle.*

For, if $[\xi, \eta]$ lies on (α), it may be shown that (σ), the bipolar of $[\xi, \eta]$, is orthogonal to a circle (γ), where (γ) bears to (α) the same relation that (α) bears to its bipolar (β). Thus,

DEFINITION 3. The *antibipolar circle* (γ) of a circle (α), with respect to the biquadratic, is the circle orthogonal to the bipolars of couples on (α).

THEOREM 4. *If (β) is the bipolar of (α), then (α) is the antibipolar of (β).*

3. The Jacobian circles. Since harmonic and orthogonal properties are invariant under transformations of the inversive group, the bipolar and antibipolar relations implicit in equations (2.4) are invariant. That is, if (β) is the bipolar of a circle (α) with respect to a biquadratic (B) and if an inversive transformation sends (β), (α), and (B) into (β'), (α'), and (B') respectively, then (β') is the bipolar of (α') with respect to (B').

Inversions are fundamental in that any inversive transformation is the product of a finite number of inversions. Consequently, we are interested in knowing whether there are any inversions which transform a given biquadratic into itself [3, 4]. If (α) is the circle associated with an inversion I_α which transforms the biquadratic (B) into itself, (B) is said to be *anallagmatic* with respect to the circle (α). If there is such a circle (α), its bipolar (β) with respect to (B) is transformed into itself by I_α. Moreover, by Definition 2, the bipolar of every couple on (β) is orthogonal to (α) and therefore transforms under I_α into itself. In fact, not only is (β) unchanged by I_α but so is every circle orthogonal to (α); hence, every couple of (β) is unchanged by I_α. Since, however, the only couples which are invariant under I_α are the couples on (α), it follows that (β) coincides with (α). Conversely, we can show that if (α) coincides with (β), then (B) is anallagmatic with respect to (α). Thus,

THEOREM 5. *A biquadratic is anallagmatic with respect to a given circle if and only if that circle is identical with its bipolar with respect to the biquadratic.*

From equations (2.4) and the condition that (α) is identical with (β), i.e., $\beta_{ik} = \mu\alpha_{ik}$ where μ is a proportionality factor, we have the coefficients of (α) given by four equations like the following:

$$(a_{11} - \mu)\alpha_{11} - a_{12}\alpha_{10} - a_{21}\alpha_{01} + a_{22}\alpha_{00} = 0. \tag{3.1}$$

These equations may be solved for the α_{ik} when μ is so chosen that D, the determinant of the coefficients a_{ik}, vanishes. Since

$$D \equiv \mu^4 + \Delta_2\mu^2 + \Delta_3\mu + \Delta_4 = 0,$$

there are four such values of μ.* The four values of μ are distinct when the biquadratic has distinct foci; two are coincident when the biquadratic has a node; three are coincident when the biquadratic has a cusp [1].

* The Δ_i are invariants of the biquadratic and bear the following relations to those given in reference [2]:

$$\Delta_2 = -I_2,\ \Delta_3 = -2I_3,\ \Delta_4 = (I_3^2 - I_4)/4.$$

Equations (3.1), with $D=0$, express the condition that (α) is orthogonal to each of four circles like

$$a_{22}z\bar{w} + a_{21}z + a_{12}\bar{w} + (a_{11} - \mu) = 0, \tag{3.2}$$

where μ is a root of $D=0$. Now, three circles (a), (b), and (c) have a common orthogonal circle whose equation is

$$\begin{vmatrix} a_{11} & a_{10} & a_{01} & a_{00} \\ b_{11} & b_{10} & b_{01} & b_{00} \\ c_{11} & c_{10} & c_{01} & c_{00} \\ 1 & -\bar{w} & -z & z\bar{w} \end{vmatrix} = 0$$

if and only if the matrix of the coefficients a_{ik}, b_{ik}, c_{ik} is of rank three. If this matrix is of rank two, the three circles are linearly dependent and have a one-parameter family of orthogonal circles; and if the rank is one, the three circles are identical and have a two-parameter family of orthogonal circles.

Thus, with respect to the circles of (3.2), for each value of μ, we consider the three possibilities:

(1) *D is of rank* 3. The circle (α) is then uniquely determined as the common orthogonal circle of three of the four circles given by (3.2). So determined, (α) is also orthogonal to the fourth circle, since $D=0$.

(2) *D is of rank* 2. The four circles of (3.2) form a coaxal system, and (α) is any circle of the orthogonal coaxal system.

(3) *D is of rank* 1. The four circles of (3.2) are identical, and (α) is any circle of the two-parameter family of orthogonal circles.

There are, therefore, four circles (α) with respect to each of which the general biquadratic is anallagmatic; and certain special biquadratics are anallagmatic with respect to one- and two-parameter families of circles. Moreover, when the rank of D is three, an inversion with respect to any one of the circles transforms each of the others, as well as the biquadratic, into itself. Therefore,

THEOREM 6. *The general biquadratic is anallagmatic with respect to four mutually orthogonal circles.*

The four mutually orthogonal circles of the theorem are called the *Jacobian circles* J_i of the biquadratic. One of them, at least, has no real trace.

4. Inversive transformations which leave a biquadratic invariant. With each Jacobian circle there is associated an inversion I_i which transforms the biquadratic into itself; hence, in the general case, there are four inversions leaving the biquadratic invariant.

Now, it is known that the product of two inversions in orthogonal circles is commutative and is a polarity. The fixed points of this polarity are the common couples of the two circles; that is, the common points of intersection if the circles intersect, or the common inverse points if the circles do not intersect.

Hence, the four inversions I_i, associated with the Jacobian circles, generate six polarities $P_{ik}=I_iI_k$ which leave the biquadratic invariant. Since, however, the Jacobian circles are mutually orthogonal, the common couples of two of them, say J_1 and J_2, are identical with the common couples of the other two, J_3 and J_4. Thus,

$$P_{12} = P_{34}, \qquad P_{13} = P_{24}, \qquad P_{14} = P_{23},$$

and the six polarities are not distinct, but are equivalent in pairs.

The product of two polarities with distinct fixed points is, in general, a homography whose fixed points are the common harmonic conjugates of the two pairs of fixed points of the component polarities. In the case at hand, however, the product of any two of the three distinct polarities P_{12}, P_{13}, P_{14} is the third. For example,

$$P_{12}P_{13} = I_1I_2I_1I_3 = I_2I_1I_1I_3 = I_2I_3 = P_{23} = P_{14},$$

since a product such as I_1I_2 is commutative, and a repeated inversion such as I_1I_1 is the identity transformation. Hence, there are three, and only three, polarities which leave the biquadratic invariant.

The product of an inversion and a polarity is, in general, an antigraphy. Here, however, nothing new is obtained for the antigraphy turns out to be one of the four inversions I_i. For example,

$$I_1P_{23} = I_1P_{14} = I_1I_1I_4 = I_4.$$

To summarize, the Jacobian circles may be found by the method of Section 3, and then the inversive transformations which leave a biquadratic invariant are known: they consist of (1) four inversions in mutually orthogonal circles, (2) three polarities, and (3) the identity transformation. These eight elements constitute the abelian group G_8 of type (1, 1, 1).

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MATRIX METHODS IN THE SOLUTION OF ALGEBRAIC EQUATIONS*

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Although the theory of solution of algebraic equations has received its rightful share of attention on the part of mathematicians in the past, and even perhaps more than its due, we must not forget that this theory gave rise to a large number of elegant developments without which mathematics, and particularly modern algebra, would not be what it is today. For this and other reasons we feel justified in making another contribution to this already highly cultivated field. By means of the method to be given here the solution of a system of algebraic equations which arise in the determination of multiple roots of algebraic equations is reduced to the study of a single algebraic equation which can be more easily manipulated. This algebraic equation is related to the apolar of a binary form so extensively treated in the literature.†

1. Roots. The algebraic equation,

$$f(x) = k_0 a_0 x^n + k_1 a_1 x^{n-1} + \cdots + k_{n-1} a_{n-1} x + k_n a_n = 0, \qquad a_0 \neq 0, \tag{1}$$

where k_i is the coefficient of x^i in the expansion of $(1+x)^n$, and the a's are real or complex numbers,‡ has a root α repeated at least r times if and only if

$$f(x) = 0, f'(x) = 0, f''(x) = 0, \cdots, f^{[r-1]}(x) = 0. \tag{2}$$

A value of x which satisfies (2) will also satisfy the equations

$$\begin{aligned}
&t_0 a_0 x^{n-r+1} + t_1 a_1 x^{n-r} + \cdots + t_{n-r+1} a_{n-r+1} = 0,\\
&t_0 a_1 x^{n-r+1} + t_1 a_2 x^{n-r} + \cdots + t_{n-r+1} a_{n-r+2} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&t_0 a_{r-1} x^{n-r+1} + t_1 a_r x^{n-r} + \cdots + t_{n-r+1} a_n = 0,
\end{aligned} \tag{3}$$

where t_i is the coefficient of x^i in the expansion of $(1+x)^{n-r+1}$. The equivalence of the systems (2) and (3) can be proved by eliminating $x^n, \cdots, x^{n-r+2}$ successively from the equations in (2).

The system (3) is a system of equations linear in $t_0 x^{n-r+1}, t_1 x^{n-r}, \cdots, 1$ with matrix of coefficients a_{n-r+1} given by

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* An address delivered before the Mathematical Association of America at the invitation of the Program Committee at Chicago, September 2, 1941.

† E. B. Elliott, Algebra of quantics, pp. 260–267. The polynomial $F_i(x)$, which will appear later in this paper, is an apolar.

‡ The a's may belong to any field k with the property that the a's in $f(x)$ are uniquely determined by $f(x)$. If the characteristic of k is greater than n the field k has this property.

$$A_{n-r+1} = \begin{Vmatrix} a_0 & a_1 \cdots a_{n-r+1} \\ a_1 & a_2 \cdots a^{n-r+2} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ a_{r-1} & a_r \cdots a_n \end{Vmatrix}.$$

Instead of the system (3) we consider the system of equations

$$\begin{aligned} a_0\xi_0 + a_1\xi_1 + \cdots + a_{n-r+1}\xi_{n-r+1} &= 0, \\ a_1\xi_0 + a_2\xi_1 + \cdots + a_{n-r+2}\xi_{n-r+1} &= 0, \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ a_{r-1}\xi_0 + a_r\xi_1 + \cdots + a_n\xi_{n-r+1} &= 0, \end{aligned} \tag{4}$$

linear in $\xi_0, \cdots, \xi_{n-r+1}$ with the matrix of coefficients A_{n-r+1}. For the system (3) to have a solution $x=\alpha$, it is necessary that $A_{n-r+1}\xi=0$ have a solution $\xi \neq 0$, where

$$\xi = \begin{Vmatrix} \xi_0 \\ \xi_1 \\ \cdot \\ \cdot \\ \xi_{n-r+1} \end{Vmatrix}.$$

We introduce the polynomial $F_{n-r+1}(x)$ given by

$$F_{n-r+1}(x) = \xi_0 - \xi_1 x + \xi_2 x^2 - \cdots \pm \xi_{n-r+1}x^{n-r+1},$$

where the ξ's are the components of the general solution of (4). Some of the coefficients in the polynomial $F_{n-r+1}(x)$ may be arbitrary in which case $F_{n-r+1}(x)$ is really a linear family of polynomials. In particular, if A_{n-r+1} has rank t, by the theory of linear equations, there are $n-r+1-t$ arbitrary components in the solution ξ of $A_{n-r+1}\xi=0$.

There is a solution $x=\alpha$ of (3) if and only if we can choose the coefficients in $F_{n-r+1}(x)$ so that

$$F_{n-r+1}(x) = t_0\alpha^{n-r+1} - t_1\alpha^{n-r}x + \cdots \pm t_{n-r+1}x^{n-r+1} = (\alpha - x)^{n-r+1}.$$

We have arrived at the following theorem where "r-fold" means "repeated at least r times."

THEOREM 1. *The equation* (1) *has an r-fold root α if and only if the coefficients in $F_{n-r+1}(x)$ can be chosen so that α is an $(n-r+1)$-fold root of $F_{n-r+1}(x)=0$.*

The determination of the roots of (1) and the multiplicities of these roots thus reduces to a study of the polynomials F_i for the various values of i.

2. Matrices and polynomials associated with (1). For each i in the range $0, 1, \cdots, n$ there is a matrix A_i of $f(x)$. Since for each pair of values of i and j the matrices A_i and A_j contain the same elements, it is to be hoped that the

ranks of these matrices are related in a simple way. Such a relation does exist and will now be given.

Let m be the smallest value of i for which $F_i(x)$ is not identically zero. We state a theorem whose proof will appear elsewhere.*

THEOREM 2. *The ranks of A_i and A_{n-i} are equal to $i+1$ for $i<m$ and to m for $i \geqq m$, $(i \leqq n/2)$.*

Thus m determines the ranks of the matrices $A_0, A_1, \cdots, A_n$. From the dimensions of the matrices, $A_0, \cdots, A_n$, it follows that

$$m \leqq 1 + n/2. \tag{5}$$

Since A_m has $n-m+1$ rows the rank of A_m is m when $n-m+1 \geqq m$; that is, $m<1+n/2$. If $m=1+n/2$, the rank of A_m is $m-1$. It follows that if m satisfies the condition, $m<1+n/2$, there is one arbitrary coefficient in $F_m(x)$. In the remaining case there are two arbitrary coefficients.

THEOREM 3. *For each $i>m$ we can choose the coefficients in $F_i(x)$ so that*

$$F_i(x) = F_m(x)g(x),$$

where $g(x)$ is the general polynomial of degree $i-m$.

By "general polynomial" we mean a polynomial with arbitrary coefficients.

We write $F_m(x)$ and $g(x)$ as $\eta_0-\eta_1x+\cdots\pm\eta_mx^m$, $\beta_0+\beta_1x+\cdots+\beta_{i-m}x^{i-m}$ respectively. We introduce M_r and L_r, where

$$M_r = \begin{Vmatrix} a_r & \cdots & a_{r+m} \\ \cdots & \cdots & \cdots \\ a_{r+n-i} & \cdots & a_{r+m+n-i} \end{Vmatrix}, \qquad L_r = \begin{Vmatrix} 0 \\ \cdots \\ 0 \\ \eta_0 \\ \cdots \\ \eta_m \\ 0 \\ \cdots \\ 0 \end{Vmatrix},$$

r zeros preceding η_0 in L_r. Now $A_iL_r=M_r\eta$. Since M_r is a minor of A_m, and $A_m\eta=0$, it follows that $M_r\eta=0$, whence $A_iL_r=0$. Thus the linear combination

$$\xi = \beta_0L_0 - \beta_1L_1 + \beta_2L_2 - \cdots \pm \beta_{i-m}L_{i-m} \tag{6}$$

of the L's is a solution of $A_i\xi=0$ for each choice of the β's. Since for each j the vector ξ is the column vector whose jth component is $(-1)^j$ times the jth coefficient in the polynomial $F_m(x)g(x)$, it follows that $F_m(x)g(x) \equiv F_i(x)$ or $F_m(x)g(x)$ is a specialization of $F_i(x)$.

* Factorization and representation, Transactions of the American Mathematical Society.

THEOREM 4. *For each i in the range,*

$$n - m + 1 \geqq i \geqq m, \tag{7}$$

the polynomial $F_i(x)$ is identically $F_m(x)g(x)$ where $g(x)$ is the general polynomial of degree $i-m$.

We need but consider the case $i \neq m$ since if $i=m$ we have $F_i(x) \equiv F_m(x)$.

By Theorem 2 the rank of A_i is m, while A_i has $i+1$ columns. It follows from the theory of linear equations that, given A_i, there are $i+1-m$ linearly independent solutions of $A_i\xi=0$. This means that there are $i+1-m$ arbitrary components in the solution ξ of $A_i\xi=0$, and thus $i+1-m$ arbitrary coefficients in $F_i(x)$. When $n-m+1 \geqq m$, we have $m<1+n/2$, whence $F_m(x)$ has one arbitrary coefficient which may be taken equal to 1. Since the general polynomial $g(x)$ of degree $i-m$ has $i-m+1$ arbitrary coefficients, in the product $F_m(x)g(x)$ there are $i-m+1$ arbitrary coefficients. That is, the vector ξ of coefficients in $F_m(x)g(x)$ given in (6) is the general linear combination of $i-m+1$ linearly independent vectors $L_0, \cdots, L_{i-m}$. By the theory of linear equations the general solution of $A_i\xi=0$ is (6) and

$$F_i(x) \equiv F_m(x)g(x). \tag{8}$$

For i not in the range (7) the identity (8) is no longer valid, although by Theorem 3 the coefficients in $F_i(x)$ can be chosen so that $F_i(x)=F_m(x)g(x)$.

3. The roots of (1), and the number m. The number m yields a bound on the multiplicities of the roots of (1) as is clear from the following theorem.

THEOREM 5. *The equation* (1) *has an r-fold root α, $r \geqq m$, if and only if $F_m(x)=0$ has α as an m-fold root.*

Since $i=n-r+1$ is in the range (7), by Theorem 4 we cannot have $F_i(x) = (x-\alpha)^i$ unless $F_m(x)=(x-\alpha)^m$. If $F_m(x)=(x-\alpha)^m$, by (8) we have $F_i(x) = (x-\alpha)^i$ when we choose $g(x)=(x-\alpha)^{i-m}$.

COROLLARY 1. *The equation* (1) *has at most one r-fold root where $r \geqq m$.*

If $m<1+n/2$ Corollary 1 follows from Theorem 5 and the property that $F_m(x)=kH(x)$, where $H(x)$ is free of arbitrary coefficients.

If $m=1+n/2$, since $2m=n+2$, we cannot have two m-fold roots.

COROLLARY 2. *If* (1) *has α as an r-fold root, $r \geqq m$, the root α is an $(n-m+1)$-fold root of* (1).

Corollary 2 is a consequence of Theorems 4 and 5.

COROLLARY 3. *The equation* (1) *has no $(n-m+1)$-fold root.*

If (1) has a root α of multiplicity at least $n-m+2$, for some $i<m$ we have $F_i(x)=(x-\alpha)^i \neq 0$, a contradiction.

COROLLARY 4. *If* (1) *has an r-fold root we have* $m \leqq n-r+1$.

If (1) has an r-fold root, we have $F_{n-r+1}(x) = (x-\alpha)^{n-r+1} \not\equiv 0$.
It can be proved that if $r \geqq n/2$ we have $m = n-r+1$.

4. Equations with only one root. In view of Theorem 1 we are concerned with equations possessing only coincident roots. The following theorem is useful in the study of these equations.

THEOREM 6. *The equation* (1) *has coincident roots if and only if* A_1 *has rank* 1.

If A_1 has rank 1, for some value α we have $a_i = -\alpha a_{i-1}$. It follows that $f(x) = a_0(x-\alpha)^n$. The converse follows readily.

In place of (1) we consider $F_i(x) = 0$. For $F_i(x) = 0$ there is the matrix

$$\left\| \begin{array}{cc} \xi_0 & \dfrac{-\xi_1}{t_1} \\ \dfrac{-\xi_1}{t_1} & \dfrac{\xi_2}{t_2} \\ \cdots & \cdots \\ \dfrac{\mp \xi_{i-1}}{t_{i-1}} & \pm \xi_i \end{array} \right\|, \tag{9}$$

corresponding to A_1 for (1), where t_j is the coefficient of x^j in the expansion of $(1+x)^i$. Since $\xi_i \neq 0$ when $F_i(x) = (x-\alpha)^i$, we have $F_i(x) = (x-\alpha)^i$ if and only if all rows of (9) are multiples of the last. This means that

$$\xi_i^{i-j-1}\xi_j = t_j\xi_{i-1}^{i-j}, \qquad j = 0, 1, \cdots, i-2, \tag{10}$$

with $\xi_{i-1} = \xi_{i-1}/t_{i-1}$. The relations (10) and $A_i\xi = 0$ obviously yield a system of equations equivalent to (3) with $i = n-r+1$.

5. Illustration. We consider (1) where

$$f(x) = x^6 - 2x^5 - x^4 + 4x^3 - x^2 - 2x + 1.$$

To avoid fractions we replace $f(x)$ by $15f(x)$. Much of the development below is given, not because it is needed to determine the roots of (1) and their multiplicities, but rather to illustrate each of the main points in the theory of earlier sections.

The matrix A_3 is given by

$$\left\| \begin{array}{rrrr} 15 & -5 & -1 & 3 \\ -5 & -1 & 3 & -1 \\ -1 & 3 & -1 & -5 \\ 3 & -1 & -5 & 15 \end{array} \right\|.$$

The rank of A_3 is 3, whence $m=3$. By Theorem 5 the equation (1) does not have a 3-fold root unless $F_3(x)=0$ has such a root. For this reason we proceed to compute $F_3(x)$. Solving $A_3\xi=0$ for ξ we obtain

$$\xi = k \begin{Vmatrix} 1 \\ 3 \\ 3 \\ 1 \end{Vmatrix},$$

whence with $k=-1$ we have $F_3(x)=(x-1)^3$. By Theorem 1 the value 1 is a 4-fold root of (1). The matrix A_4, given by

$$\begin{Vmatrix} 15 & -5 & -1 & 3 & -1 \\ -5 & -1 & 3 & -1 & -5 \\ -1 & 3 & -1 & -5 & 15 \end{Vmatrix}$$

has rank 3 in view of Theorem 2. By Theorem 4 we have $F_4(x)\equiv(x-1)^3(\beta_0+\beta_1 x)$, where the β's are arbitrary. Thus the solution ξ of $A_4\xi=0$ is given by

$$\xi = \beta_0 \begin{Vmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 0 \end{Vmatrix} - \beta_1 \begin{Vmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \end{Vmatrix}.$$

Solving $A_5\xi=0$, where

$$A_5 = \begin{Vmatrix} 15 & -5 & -1 & 3 & -1 & -5 \\ -5 & -1 & 3 & -1 & -5 & 15 \end{Vmatrix},$$

we obtain $\xi_1=\xi_2-2\xi_4+5\xi_5$, $5\xi_0=2\xi_2-\xi_3-3\xi_4+10\xi_5$. We take $\xi_5=-1$. Solving the two relations between the ξ's, just obtained, with (10), we derive a quartic equation E in ξ_4. By Theorem 3 we can restrict the ξ's so that $F_5(x)=(x-1)^5$, whence E has the solution $\xi_4=-5$. The other root of E is $\xi_4=5$, in which case $F_5(x)=(x+1)^5$. By Theorem 1 the equation (1) has -1 as a double root. Thus the roots of (1) are 1, 1, 1, 1, -1, -1.

TOPOLOGY OF THE TWO-BODY PROBLEM*

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1. Introduction. The differential equations of the two-body problem of celestial mechanics can be written as four first order equations,

$$dx_i/dt = X_i(x_1, x_2, x_3, x_4), \qquad (i = 1, 2, 3, 4),$$

which represent a flow in 4-space. If the energy constant is fixed, the trajectories fill out a three-dimensional manifold in the 4-space. This manifold we term the *energy phase space.*

It is the purpose of the present paper to answer the following questions: 1. What is the topological† structure of the energy phase space? 2. If the area constant is also fixed, what is the topological structure of the two-dimensional subspace of the energy phase space thereby determined? (This surface will be called the *energy-area phase space.*) 3. As the area constant varies, how do the surfaces of question 2 fill out the energy phase space? 4. How do the individual trajectories fill out each energy-area phase space? 5. What is the topological structure of the whole family of trajectories in the energy phase space?

Throughout the energy constant will be assumed to be negative, so that the paths are elliptical.

Poincaré was the first to consider such questions. Recently Birkhoff has considered similar questions for the three-body problem and for general dynamical systems. Exact references are given in the bibliography.

The answers to the above questions will be given in detail below. In brief they are as follows: 1. The energy phase space has the structure of 3-space minus a line. 2. The energy-area phase space has (in general) the structure of a torus. 3. Imagine the paraboloid of revolution $x^2+y^2=z-1$ to be cut by the family of all planes through the y-axis. This gives a curve-family consisting of ellipses plus one parabola and two points as limiting cases. Imagine the paraboloid mapped homeomorphically on the half-plane $y>0$ of the yz-plane, so that the above curve family becomes one filling the half-plane. (See Fig. 2.) Now imagine the half-plane rotated about the z-axis. The curves of the family sweep out surfaces which are in general of the type of the torus. This family of surfaces is the answer to question 3. 4. When the energy-area space is not degenerate and is hence like a torus, the family of trajectories is the same as a family of "parallel" circles on a torus. 5. Assume one curve of the family in the yz-plane in the answer to question 3 to be the positive y-axis. As the yz-plane is rotated each curve of the family in $z\neq 0$ sweeps out a surface like a torus. Now imagine each point of each such curve to move and make one complete revolution of the curve as the curve makes one complete rotation. The revolution is to follow the posi-

* Presented to the American Mathematical Society in Chicago, September 5, 1941.

† A topological property means one invariant under a homeomorphism. A homeomorphism is a transformation which is one-to-one and continuous in both directions. See, for example, Seifert and Threlfall, Lehrbuch der Topologie, Leipzig, 1934, especially Chapter I.

tive orientation of the curve for $z>0$ and the negative orientation for $z<0$. Thus a family of trajectories is formed on the family of torus-like surfaces. The remainder of the trajectories are two circles, obtained from the two point-curves and the family of rays from the origin of the xy-plane. This gives a rough picture of a model of the family of trajectories in the energy phase space.

2. The differential equations and integrals. If the origin of coordinates is taken at one of the two bodies and units are properly chosen, the differential equations for the motion of the second body can be written:

$$d^2x/dt^2 = -x/r^3, \qquad d^2y/dt^2 = -y/r^3, \qquad r = \sqrt{x^2+y^2}. \tag{1}$$

(See Wintner [3], p. 178.)

These have integrals

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\frac{2}{r}+c_1, \tag{2}$$

$$x\frac{dy}{dt}-y\frac{dx}{dt}=c_2, \tag{3}$$

where c_1 is termed the *energy constant*, c_2 the *area constant*.

We rewrite (1) as four first order equations thus:

$$dx/dt = z, \qquad dy/dt = w, \qquad dz/dt = -x/r^3, \qquad dw/dt = -y/r^3. \tag{4}$$

These represent a vector field in 4-dimensional $xyzw$-space. For $x=y=0$ the differential equations break down. We shall therefore delete the plane $x=y=0$ from the 4-space. In the remaining space (4) then defines a non-singular family of trajectories.

If (4) is used, (2) and (3) become

$$z^2+w^2 = 2/r + c_1, \tag{5}$$

$$xw - yz = c_2. \tag{6}$$

For fixed c_1, (5) represents a 3-dimensional variety E in $xyzw$-space such that every trajectory of (4) which meets E lies wholly in E. The same holds for the variety A defined by (6), when c_2 is fixed. We term E the *energy phase space* and A the *area phase space*. If both c_1 and c_2 are fixed, the intersection of E and A is a two-dimensional variety EA which is again built up of whole trajectories.

Equations (1) or (4) have a further integral, which is usually simplified by means of (2) and (3). In its general form the third integral can be written

$$\theta - \cos^{-1}\left(\frac{r-(xw-yz)}{r\sqrt{(xw-xz)^2(z^2+w^2-2/r)+1}}\right) = c_3 \tag{7}$$

where θ is the polar coördinate angle in the xy-plane.

If c_1 and c_2 are fixed, (7) reduces to

$$r = \frac{c_2^2}{1 - \sqrt{1 + c_1 c_2^2}\cos(\theta - c_3)} \tag{8}$$

which shows that the solutions project on the xy-plane as conic sections. The eccentricity e satisfies

$$e^2 = 1 + c_1 c_2^2. \tag{9}$$

We shall assume throughout that c_1 is negative, so that $e<1$ and the motion is elliptical. Furthermore we assume that c_2 satisfies

$$c_2^2 < -1/c_1 \tag{10}$$

in order to obtain real solutions.

If $c_2=0$, (8) breaks down, but (7) gives $\theta=c_3$, so that the solutions project on straight lines. Furthermore it follows from (2) that, since c_1 is negative, the path in the xy-plane must lie in the region $0<r\leqq -2/c_1$.

3. Topology of the energy phase-space and of the area phase-space.

THEOREM 1. *E is homeomorphic with 3-space minus a straight line.*

Proof: Since the points of the zw-plane are deleted from the 4-space, the transformation

$$r' = 1/r, \qquad \theta' = \theta, \qquad z' = z, \qquad w' = w, \tag{11}$$

which is simply an inversion in the unit circle in the xy-plane, is a homeomorphism of the deleted 4-space on itself. (5) becomes

$$z'^2 + w'^2 = 2r' + c_1. \tag{5'}$$

Now since θ' does not appear in (5'), the hypersurface which it represents is a hypersurface of revolution. The 2-surface from which it is obtained by revolution can be found by plotting (5') in $z'w'r'$-space, where r' is now a rectangular coordinate satisfying $r'\geqq 0$. But in that space (5') represents a paraboloid of revolution not meeting the plane $r'=0$. Thus in 4-space (5') represents a hypersurface obtained from the paraboloid by revolution about the $z'w'$-plane. Since the paraboloid is homeomorphic with the interior of a half-plane, it follows that the hypersurface is homeomorphic to one obtained by revolving the interior of a half-plane about its boundary line. This gives a 3-space minus a straight line as desired.

In terms of topological products* the hypersurface (5') is paraboloid$\times$circle

* The topological product of two geometrical objects α and β can here be defined as follows: Suppose α to lie in $(x_1, \cdots, x_n)$-space and β to lie in $(y_1, \cdots, y_m)$-space. The product $\alpha\times\beta$ consists of those points of $(x_1, \cdots, x_n, y_1, \cdots, y_m)$-space whose x-coordinates give a point of α and whose y-coordinates give a point of β. Any object homeomorphic with $\alpha\times\beta$ as thus defined is also termed the topological product $\alpha\times\beta$. Examples: square=line segment$\times$line segment; torus =circle$\times$circle.

or line×line×circle, where line = infinite straight line.

COROLLARY. *If $c_2 \neq 0$, A is homeomorphic with 3-space minus a line.*

Proof: First note that for $c_2 \neq 0$, A contains none of the deleted points of $x=0$, $y=0$.

Suppose $c_2 < 0$. By rotations in the xw-plane and yz-plane (6) becomes

$$x''^2 - w''^2 - y''^2 + z''^2 = 2c_2. \tag{12}$$

Let ρ, ϕ be polar coordinates in the $y''w''$-plane. (12) becomes

$$x''^2 + z''^2 = \rho^2 + 2c_2. \tag{13}$$

The transformation

$$x''' = x'', \qquad z''' = z'', \qquad \rho''' = \rho^2, \qquad \phi''' = \phi$$

is a homeomorphism of the 4-space. (13) becomes

$$x'''^2 + z'''^2 = \rho''' + 2c_2. \tag{14}$$

Since $2c_2 < 0$, this is the same as (5′), and the corollary follows as above.

The same method applies if $c_2 > 0$, but it breaks down if $c_2 = 0$. If $c_2 = 0$, it can be shown that A (with the plane $x=0$, $y=0$ excluded) has still the same topology.

4. The energy-area phase space. Suppose now that c_1 and c_2 are fixed ($c_1 < 0$). Consider then the intersection EA of E and A.

THEOREM 2. *If $0 < c_2^2 < -1/c_1$, then EA is homeomorphic with the surface of a torus. If $c_2 = 0$, EA is homeomorphic with the interior of an annulus. If $c_2^2 = -1/c_1$, EA is homeomorphic to a circle.*

Proof: Use the coordinates r', θ', z', w' of §3. The equations of EA are

$$\begin{aligned} z'^2 + w'^2 &= 2r' + c_1, \\ w' \cos\theta' - z' \sin\theta' &= c_2 r'. \end{aligned} \tag{15}$$

Suppose now θ' fixed, so that we can plot in $r'z'w'$-space. Then (15) gives the intersection of a paraboloid and a plane. The plane passes through the origin and makes an angle of arc sin $c_2/\sqrt{1+c_2^2}$ with the r'-axis. Its line of intersection with the $w'z'$-plane makes an angle of θ' with the z'-axis. Thus as θ' varies from 0 to 2π the plane starts from the position $w' = c_2 r'$ and makes a complete revolution, in the direction from the positive z'-axis to the positive w'-axis, about the r'-axis. Since the paraboloid is one of revolution about the r'-axis, the type of intersection will be independent of θ'. For $\theta' = 0$ it reduces to the curve,

$$z'^2 + w'^2 = 2r' + c_1, \qquad w' = c_2 r', \tag{16}$$

whose projection on the $z'w'$-plane is

$$z'^2 + \left(w' - \frac{1}{c_2}\right)^2 = \frac{1 + c_1 c_2^2}{c_2^2}. \tag{17}$$

Thus, if $1+c_1c_2^2>0$ and $c_2\neq 0$, the intersection is an ellipse. As θ' varies from 0 to 2π the ellipse rotates about the $z'w'$-plane and finally returns to its original position. This gives a surface with the structure of a torus. In fact we can immediately introduce coordinates (θ', ϕ) on the torus, where ϕ denotes position on the ellipse (15), for fixed θ'. The coordinate ϕ can be specifically defined thus: Project the ellipse on the $z'w'$-plane; let Q^* be the center of the image and O^* the origin of the $z'w'$-plane. Assume the $z'w'$-plane is oriented so that angles from

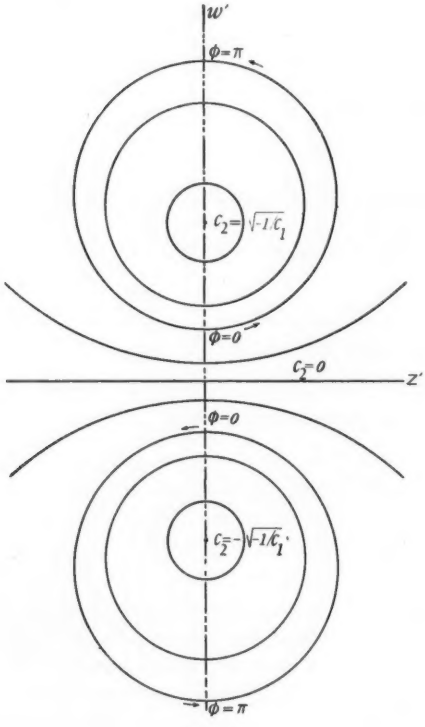

FIG. 1

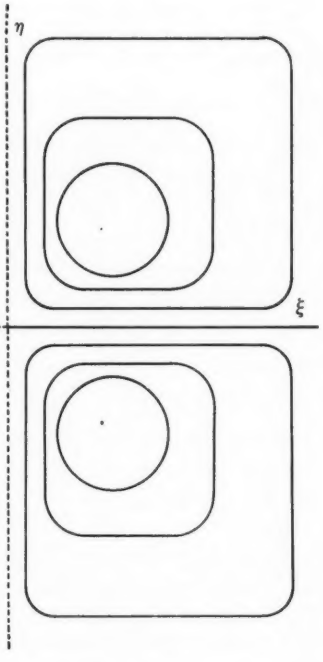

FIG. 2

the positive z'-axis to the positive w'-axis are counted positive. Let ϕ be the angle from Q^*O^* to Q^*P^* for any point P^* on the image. This gives the coordinate ϕ for the point P above P^*.

If $c_2=0$, the ellipse (15) becomes the parabola

$$z'^2 = 2r' + c_1, \quad w' = 0.$$

As θ' varies, this generates the topological product of a parabola with a circle. Since a parabola is homeomorphic with an open interval, the surface here is homeomorphic with the interior of an annulus.

If $1+c_1c_2^2=0$, the ellipse (15) degenerates to the point $(0, 1/c_2, 1/c_2^2)$. As θ' varies this generates a simple closed curve. (Note that this occurs for $c_2=\sqrt{-1/c_1}$ or $c_2=-\sqrt{-1/c_1}$.) Thus Theorem 2 is established.

5. Decomposition of the energy phase space. The trajectories (4), for fixed c_1, form a curve-family filling E. For c_2 fixed, the trajectories form a curve-family filling EA. As c_2 varies, the surfaces EA sweep out E. Thus they decompose E into surfaces, each of which is a collection of trajectories.

The range of c_2 is from $-\sqrt{-1/c_1}$ to $\sqrt{-1/c_1}$. At the two extremes EA reduces to a closed curve. In between EA is topologically a torus, except for $c_2=0$, when EA is homeomorphic with the interior of an annulus. Now E itself is homeomorphic with 3-space minus a line. The question to be considered is how the family of surfaces EA fills out E.

This is immediately answered by considering the picture of §4. There we held θ' fixed and considered the locus in $r'w'z'$-space. If c_2 varies over its interval $[-\sqrt{-1/c_1}, \sqrt{-1/c_1}]$, the paraboloid is decomposed as indicated in Figure 1. In that figure $\theta'=0$ and the paraboloid is viewed from along the r'-axis. The picture changes with varying θ' only by a rotation about the origin. In the figure the ellipses are drawn as circles, but from a topological point of view this is no specialization.

As θ' varies from 0 to 2π the configuration pictured is multiplied topologically by a circle. The resulting family of surfaces can be pictured topologically in three-space if we first map the $w'z'$-plane of Figure 1 on a half-plane $\xi>0$ of the $\xi\eta$-plane. This gives the picture of Figure 2. Now rotate the family of curves pictured about the η-axis. This gives a family of tori (with degenerate cases) filling three-space minus a line as desired.

6. Trajectories on each torus EA. It now remains to consider how the trajectories are distributed on each torus. The trajectories have projections on the xy-plane given by (8). For fixed c_1, c_2, c_3 with $c_1<0$, $c_2\neq 0$ (8) represents an ellipse. If c_3 is allowed to vary, the ellipse is rotated about the origin.

If we make the transformation (11), so that the ellipse is inverted in a circle, (8) becomes

$$r' = \frac{1-\sqrt{1+c_1c_2^2}\cos(\theta'-c_3)}{c_2^2} \tag{8'}$$

which is the equation of a limaçon. The important properties of this curve for the following are that r' is a single-valued function of θ', and that as θ' goes from 0 to 2π, r' has exactly two extrema, one maximum and one minimum.

Now (8') is the projection of a trajectory which is on a torus EA, given by (15). The part of EA in a 3-space $\theta'=$constant is an ellipse, as in Figure 1. For each θ', (8') gives just one r'; hence there is exactly one point $P_{\theta'}$ of the trajec-

tory on each ellipse $\theta' =$ constant of (15). As θ' varies, $P_{\theta'}$ must then vary on the torus. The trajectory can thus be written as

$$\phi = f(\theta') \qquad (f(\theta' + 2\pi) \equiv f(\theta') \bmod 2\pi). \tag{19}$$

Furthermore all the trajectories on the same torus will have the same equation as (19), except for a translation of θ' ,as follows from (8'); that is, the family of trajectories is given by

$$\phi = f(\theta' + \alpha) \qquad 0 \leqq \theta' \leqq 2\pi, \qquad 0 \leqq \alpha \leqq 2\pi. \tag{20}$$

A direct study of the curve (8') on the torus shows that the function f is monotone strictly decreasing if $c_2 > 0$ and monotone strictly increasing if $c_2 < 0$. Hence

$$f(\theta' + 2\pi) - f(\theta') = 2k\pi, \qquad |k| \geqq 1,$$

where k is an integer and $kc_2 < 0$ ($c_2 \neq 0$). If $|k|$ were greater than 1, r' would have more than one maximum, since r' has a maximum every time ϕ crosses a value $(2n+1)\pi$ ($n = 0, \pm 1, \pm 2, \cdots$). But r' has just one maximum, and hence $|k| = 1$. Thus each curve (20) has an increase (if $c_2 < 0$) or a decrease (if $c_2 > 0$) in ϕ of 2π for every circuit.

From the monotone character of f we conclude that the family (20) can also be written

$$\theta = g(\phi) - \alpha, \qquad 0 \leqq \phi \leqq 2\pi, \qquad 0 \leqq \alpha \leqq 2\pi. \qquad (\theta' = \theta). \tag{21}$$

Such a family of curves on the torus is homeomorphic with a family obtained from the family of generators of a finite cylinder $x^2 + y^2 = 1$, $0 \leqq z \leqq 1$ by identifying boundary points with the same xy-coordinates. The appearance of a twist of 2π has to do solely with the way the torus is imbedded in E.

7. The degenerate cases of EA. If $c_2 = 0$, EA is homeomorphic to the interior of an annulus. In this case the trajectories have projections $\theta' =$ constant on the $x'y'$-plane. The part of EA above a line $\theta' =$ constant is a parabola such as (18). It follows that the parabola itself is the trajectory. The family of trajectories is thus homeomorphic with the family of lines $\theta' =$ constant of the annulus $1 < r < 2$.

If $c_2 = \pm\sqrt{-1/c_1}$, EA reduces to a closed curve. In each of these two cases (8') reduces to a circle. Thus the two closed curves EA are trajectories, each of which projects on a circle.

8. The family of trajectories in the energy phase space. We have seen how E is decomposed into tori and how the trajectories are placed on each torus. It remains to give a picture of the whole collection of trajectories in E. This can be seen from Fig. 2. Imagine that figure rotated around the η-axis in a $\xi\eta\zeta$-space. As each ellipse traces out its torus, the trajectories through it trace out the paths given as above by $\theta = g(\phi) - \alpha$, where θ denotes the angle through which the figure has been rotated. The function g will depend on the ellipse chosen, which in turn depends on c_2. Thus we write $\theta = g(\phi; c_2) - \alpha$ as the equation of the family.

For $c_2=0$, the trajectories are given by the rays from the origin in the $\xi\zeta$-plane. For $c_2=\pm\sqrt{-1/c_1}$ the trajectories are two circles, one above and one below the $\xi\zeta$-plane. Since the trajectories must fit together as do the solution of a differential equation, we must have

$$\lim_{c_2\to 0,\phi\to 0}\frac{\partial\theta}{\partial\phi}=0, \qquad \lim_{c_2\to\pm\sqrt{-1/c_1}}\frac{\partial\theta}{\partial\phi}=\pm\infty.$$

It can be proved that this description completely characterizes the family from a topological point of view; *i.e.*, if two such families, F_1 and F_2, are given, both of which satisfy the above description, then there is a homeomorphism of $\xi\eta\zeta$-space (minus the η-axis) on itself which transforms F_1 onto F_2.

9. Introduction of time parameter. Thus far in considering the trajectories we have neglected the time parameter. But work of Whitney* has shown that from the topological point of view the parameter is almost completely determined by the family of trajectories.

One point of interest however is how the directions of motion orient the family of trajectories. For $c_2>0$, the motions are known to be direct ($d\theta/dt>0$) and for $c_2<0$ the motions are retrograde ($d\theta/dt<0$) This implies that in the above picture of the family, the motions for $c_2>0$ are in the direction of increasing θ, those for $c_2<0$ are in the direction of decreasing θ, and those for $c_2=0$ are away from the origin.

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2. H. Poincaré, Les Methodes Nouvelles de la Mécanique Céleste, Vol. 1, 1892, Vol. 2, 1893, Vol. 3, 1899.

3. A. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton, 1941.

* See Hassler Whitney, Regular families of curves, Annals of Mathematics, Vol. 34, 1933, pp. 244–270, esp. pp. 269–270.

DISCUSSIONS AND NOTES

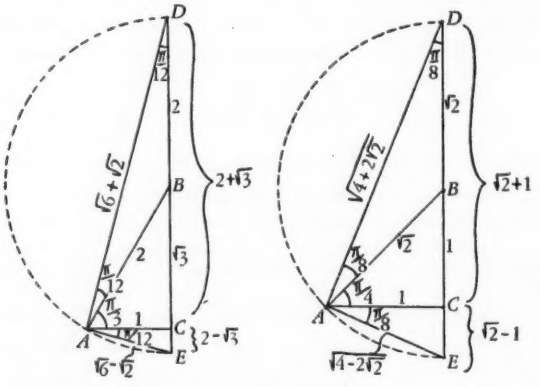
EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

VALUES OF THE TRIGONOMETRIC RATIOS OF $\pi/8$ AND $\pi/12$

H. L. DORWART, Washington and Jefferson College

Although every textbook in trigonometry begins with the determination of the exact values of the trigonometric ratios of $\pi/3$, $\pi/4$ and $\pi/6$ from appropriate triangles, all texts that the writer has seen reserve the exact values of $\pi/8$ and $\pi/12$ until after the functions of sum and difference and half angles have been derived. Since the exact values of the ratios of these latter angles can be



just as useful at the beginning of a trigonometry course as those of the former, the following simple construction* may be of some interest.

In each figure, starting with the basic triangle ABC, a semicircle of radius AB is described on BC (extended) with B as a center. ABD is thus an isosceles triangle and angle DAE is a right angle.

From the right triangle DAE, we have

$$\sin\frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4},\qquad \sin\frac{\pi}{8}=\frac{\sqrt{2-\sqrt{2}}}{2},$$

$$\cos\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4},\qquad \cos\frac{\pi}{8}=\frac{\sqrt{2+\sqrt{2}}}{2},$$

* The idea of making such a construction was suggested by one of my students, Mr. R. L. Mills.

from the right triangle ACE

$$\tan\frac{\pi}{12} = 2 - \sqrt{3}, \qquad \tan\frac{\pi}{8} = \sqrt{2} - 1,$$

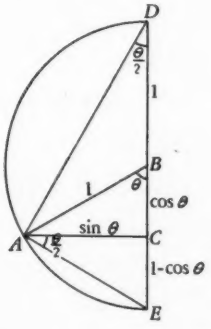
$$\sec\frac{\pi}{12} = \sqrt{6} - \sqrt{2}, \qquad \sec\frac{\pi}{8} = \sqrt{4 - 2\sqrt{2}},$$

and from the right triangle ACD

$$\cot\frac{\pi}{12} = 2 + \sqrt{3}, \qquad \cot\frac{\pi}{8} = \sqrt{2} + 1,$$

$$\operatorname{cosec}\frac{\pi}{12} = \sqrt{6} + \sqrt{2}, \qquad \operatorname{cosec}\frac{\pi}{8} = \sqrt{4 + 2\sqrt{2}}.$$

Note by the Editor. Professor Dorwart's construction suggests a simple derivation of the half-angle formulas for angles less than 180°. It is used thus occa-



sionally in texts in trigonometry. The accompanying figure is self-explanatory, and we see that

$$\tan\theta/2 = AC/CD = \sin\theta/(1 + \cos\theta)$$
$$= EC/AC = (1 - \cos\theta)/\sin\theta.$$

The other half-angle formulas are easily obtained from these. R. J. W.

A SIMPLE GEOMETRICAL PARADOX

J. L. Coolidge, Harvard University

1. The general form for the equation of a quadric surface is

$$\Sigma a_{ij}x^i x^j = 0, \qquad i, j = 1, \cdots, 4, \qquad a_{ij} = a_{ji}.$$

Here we have ten independent coefficients a_{ij}. If we require the surface to pass through a given point, we impose a linear homogeneous condition on these

coefficients. Hence we may pass a single surface through nine arbitrary points, or, what amounts to the same thing,

Ten general points in space do not lie on a quadric surface.

2. A quadric surface may be generated, after the procedure of Seydewitz, by the intersection of the lines of a bundle with the planes of a second bundle projectively related to it by a correlation. The surface will pass through the centers of these two bundles. A correlation between two bundles is essentially the same thing as a correlation between two planes. This may be expressed

$$\Sigma b_{ij}x^iy^j = 0, \qquad i, j = 1, 2, 3, \qquad |b_{ij}| \not\equiv 0.$$

This tells us that the point (y) correspond to a line through a point (x). There are nine independent coefficients b_{ij}, hence we may set up a correlation in which eight general points correspond to lines through eight other general points. Or we may set up a correlation between two bundles so that eight general lines of one correspond to planes through eight general lines of the other.

Now let A, B, $C_1, \cdots, C_8$ be ten general points of space. We may set up such a correlation that the eight lines $AC_1, \cdots, AC_8$ correspond to planes through the eight lines $BC_1, \cdots, BC_8$.

Ten general points in space lie on a quadric surface.

Which is right?

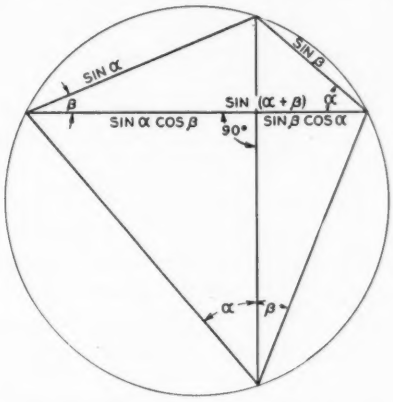
THE ADDITION FORMULAS IN TRIGONOMETRY

A. S. HOUSEHOLDER, University of Chicago

Cauchy's derivation of the addition formulas in trigonometry, given by Hobson in his *Treatise* and presented recently by McShane, is general and shorter than the standard proof given by most texts. The proof suggested by the accompanying figure (the circle has unit diameter but the figure is otherwise self-explanatory) is admittedly not general and moreover requires a preliminary lemma to the effect that the length of a chord in a circle of unit diameter is equal to the sine of the subtended inscribed angle. Nevertheless, this lemma follows almost immediately from standard high school geometry theorems relating to inscribed angles, and is otherwise useful in providing a simple proof for the law of sines. Once the lemma is established, the addition formula for the sine is immediately evident from this figure, for the case α, β, and $\alpha+\beta$ numerically less than 180°, and no algebraic manipulation is required. The same figure is easily adapted to the cosine formula.

This proof can hardly be new. Its relation to Ptolemy's theorem is too close, and that these formulas are special cases of Ptolemy's theorem is well known. But it does not seem to have found its way into the standard texts, though even if a single general proof is to be demanded it should be pedagogically a worthy adjunct to this.

Note by the Editor. Professor E. F. Beckenbach also has pointed out that McShane's method of deriving the addition formulas is essentially due to



Cauchy, and that this derivation, along with a history of these formulas, appears in Enciclopedia delle Matematiche Elementari, Milano, 1937, vol. 2, part 1, pp. 551–552. R. J. W.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

College Algebra. Second Edition. By H. P. Pettit and P. Luteyn. New York, John Wiley and Sons, Inc. 1941. 14+247 pages. $1.90.

The first edition of this fine text was reviewed in the MONTHLY, vol. 40, 1933, pp. 288–289, with but few criticisms.

The present edition still embodies the noteworthy feature of introducing the student to new materials immediately, such as functions, graphs, and the summation notation, thereby reviewing indirectly a considerable amount of elementary algebra. This is followed by the usual topics of a college algebra text, but arranged so that the ideas and notations are readily accessible for later use and developing in the student "a greater capacity for independent thought." A chapter on probability, omitted in the first edition, has been included and makes for a well-rounded text.

Especially good are the many problems which are of a practical value and related to the various sciences. These in themselves provide the student with

much worth-while information and integrate his experiences in mathematics with those of the natural sciences. Whole chapters are presented on problem solving, and answers are given for the odd-numbered examples.

Few errors were noted. Minor mistakes were found on pages 27, 57, 119, 129, and 246, the most confusing being the writing of triangle *OMA* on page 149, instead of triangle *ONA*.

The book is well written, and the typography is extremely pleasing to the eye, beginning with an attractive bookplate and concluding with exponential and logarithmic tables.

R. A. HARRISON

Intermediate Algebra. By Neil McArthur and Alexander Keith. London, Methuen and Company, 1942. 10+356 pages. Price 8/6.

As stated by the authors "This book is intended for use in school and first-year university classes composed of students who are not (or not yet) specializing in mathematics."

The introduction, which is a brief review of the number system, is well done.

As compared with United States texts on college algebra, this work assumes a somewhat better preparation than is given in most of the secondary schools in the United States. Most of our college algebras include a chapter on Interest and Annuities which is not included in this text. They also usually give solution of determinants of any order while only second and third order determinants are considered here. Also the approximation of irrational roots of equations of higher degree are usually considered while this is not included. As a whole, however, this treatment of the subject is more complete than is usual in similar texts in the United States. Their treatment of inequalities and surds is also more exhaustive.

In addition to the usual topics discussed in our texts the following are included in this book: work on symmetrical functions and reciprocal equations; the treatment of sequences and series includes Arithmetico-Geometrical series and series in which the nth term is a polynomial in n; an excellent discussion of graphs is given: symmetry and continuity are discussed, also two-point and three-point contact of curves, and maxima and minima; the idea of derivative is introduced in connection with the slope of the tangent to the curve at a point, and the equation of this tangent; the slope is also determined by the method of limits. Also, the multinomial theorem, that is $(a+b+c+\cdots)^n = \sum_{r=0}^{n} {}_nC_r a^r (b+c+\cdots)^{n-r}$ is given in connection with the binomial theorem; arithmetical series of higher order, and expansion in power series are considered.

The book seems to me to be well written. The explanations are clear and adequate; illustrative examples are sufficient in number; and there is a wealth of examples and problems for class use.

SARA L. NELSON

The Mathematics of Finance. By Llewellyn Rood Perkins and Ruth Marion Perkins. New York, John Wiley and Sons, Inc., 1941. 20+321 pp. $3.25.

This is a well written textbook, covering in a precise and lucid manner the conventional fields of the so-called mathematics of finance. According to the preface, it is directed to the student who already has a good grounding in algebra. Hence, with the exception of a discussion of rounded multiplication and division, no explanations of algebraic and other techniques are given. Some instructors, therefore, may miss the usual explanation of such elementary questions as logarithms or an appendix of log-tables. It should be stressed, however, that a decided asset of the book is a very complete set of tables which—especially in the field of life insurance—go far beyond similar textbooks.

The authors duly emphasize points of connection with actual financial practice and it seems to the reviewer that they go successfully farther in this direction than many others (*e.g.*, in chapter VII on building and loan associations).

As far as the method is concerned, it is noteworthy for its rather unusual employment of graphical representation, somewhat similar to the manner of exposition more commonly found in economic textbooks. This should prove helpful, especially to students not primarily trained in mathematics and who, otherwise, might have difficulties to grasp the analytical pattern of thought directly.

For a revised edition, the following minor suggestions might be made. The increasing mechanization of business analysis might justify the addition, in an appendix, of a short paragraph on the appropriate arrangement of data for and the use of calculating machines (as in most textbooks of economic statistics). Perhaps also, a hint at the technique of handling special slide rules, and their adaptation to some of the standard problems in the mathematics of finance would be useful. Many students are surprisingly unfamiliar with such manipulations. While admiring the technical excellence of this as well as of most American textbooks as compared with much less elaborately equipped European books, the present reviewer cannot suppress the suggestion to print tables on an extended hinge. This would enable the student to have the table in sight while reading the text and would also make it possible to print some of the larger tables in a more comprehensive way.

The book should certainly be a valuable help to the teacher as well as to the student.

J. E. MORTON

The Trisection Problem. By R. C. Yates. Baton Rouge, La., The Franklin Press, Inc., 1942. 68 pages. $1.00.

This little book has for its purpose the explanation of the trisection problem, and to show why it is impossible to solve it by straight edge and compasses. No knowledge of mathematics beyond plane geometry, elementary algebra, and the rudiments of trigonometry and of plane analytic geometry is presupposed.

It is first shown that the solution of the problem depends on that of a reduced cubic equation $x^3-3x-2a=0$ and that a construction based on these instruments leads only to certain values of a. This part is well done. It is then shown that an angle of the form $2\pi/n$, n an integer, can or cannot be trisected by straight edge and compasses according as n is not or is a multiple of 3. It is not shown under what restrictions on n such angles can be constructed.

The second chapter discusses a number of curves which intersect a given circle in the required points and adds the remark that infinitely many such curves exist, none of which can be constructed by the instruments allowed. This chapter is followed by one on mechanical trisectors, based on linkages, which is rather extensive. This is followed, in turn, by one on approximations. A brief historical note is added to many sections and a bibliography is given of works referred to in them. It makes no claim at being complete.

The back fly-leaf is a museum of newspaper clippings on "solutions." May this book do its part to quell the flood of trisectors.

VIRGIL SNYDER

NEW BOOKS RECEIVED

Elementary Mathematics in Artillery Fire. By J. M. Thomas. (With Tables prepared by Vincent H. Haag.) New York and London, McGraw-Hill Book Company, Inc., 1942. 11+256 pages. $2.50.

The Trisection Problem. By R. C. Yates. Baton Rouge, La., The Franklin Press, Inc., 1942. 68 pages.

Plane and Spherical Trigonometry. By P. R. Rider. New York, The Macmillan Company, 1942. 7+180 pages. $1.75.

An Introduction to Analytic Geometry and Calculus. By T. K. Raghavachari. Madras, Humphrey Milford and Oxford University Press, 1941. 8+192 pages. Rs 2.

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT AND J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Brown University, Providence, R. I.

THE NIMATRON

E. U. CONDON, Westinghouse Electric and Manufacturing Co.

The Nimatron is a machine which is very skillful at playing the game of Nim. Unlike other mathematical machines, the Nimatron serves no other useful purpose than to entertain, unless it be to illustrate how a set of electrical relays can be made to make a "decision" in accordance with a fairly simple mathematical procedure.

The machine was built in the spring of 1940 and was exhibited at the Westinghouse Building of the New York World's Fair, where it played more than 100,000 games and won 90,000 of them. Most of its defeats were at the hands of the exhibit attendants as demonstrations to folks who, after numerous trials, became convinced that the machine couldn't be beaten. Now it belongs to the scientific collections of the Buhl Planetarium in Pittsburgh. It was invented by

two members of the staff of the Westinghouse Research Laboratories during a long lunch hour, and considerably improved by one of the engineers of the switchgear department of the East Pittsburgh Works of the Westinghouse Electric and Manufacturing Company, where it was designed and built. The Nimatron made its last "personal appearance" at the convention of the Allied Social Science Associations in New York City under the sponsorship of the American Statistical Association and the Institute of Mathematical Statistics.

Full details of the circuit diagrams together with a detailed description are given in U.S. Patent Number 2,215,544, obtainable from the U.S. Office in Washington, D.C.

Editorial Note. The theory of the game of Nim is due to C. L. Bouton, *Annals of Mathematics*, ser. II, vol. 3, 1901, p. 35. It was recently discussed by D. P. McIntyre, this MONTHLY, vol. 49, 1942, p. 44.

CLUB REPORTS 1940–41

PI MU EPSILON NATIONAL CONVENTION

The triennial convention of the chapters of *Pi Mu Epsilon* was held in Lamberton Hall of Lehigh University on Thursday, January 1, 1942 at 12:30 P.M. Twenty-nine members and friends representing ten chapters were present. Following the luncheon a short business meeting was held with Professor Shook of Lehigh presiding. Professor Shook called the roll by chapters and members present were introduced. A partial report for the nominating committee was made by Professor Owens. Dr. Moses Richardson of the Department of Mathematics of Brooklyn College addressed the convention on *Some aspects of freshman mathematics*, listing some of the difficulties which freshmen experience and making suggestions as to their solution. The convention closed at 2:00 P.M. with a vote of thanks to the Department of Mathematics of Lehigh University and Professor Cutler, chairman of the committee on arrangements.

Mathematics Club, Massachusetts Institute of Technology

Five meetings were held during the year at which the following topics were discussed by members from the faculty of the departments at the Institute: *The applications of the theory of waves* by Professor Morse of the physics department, *Nomographic charts* by Mr. Adams of the graphics department, *The differential analyser* by Professor Taylor of the electrical engineering department, *Industrial statistics* by Mr. Hermistone, and *Foundations of statistics* by Professor Wadsworth of the mathematics department. Officers were: President, Charles Papas; Vice-President, O. K. Smith; Secretary-Treasurer, Earl Singleton; Program Manager, Marvin Epstein.

Mathematics Club, New Jersey State Teachers College at Montclair

Semi-monthly meetings were held throughout the year and the following topics were presented: *Books, old and new* by Professor V. S. Mallory, *Complex numbers* by Henry Hausdorff, *Fun in mathematics* by Shirley Stamer, *Mathematical puzzles* by Barbara Stauffer, *Paper folding* by Robert Maurer, *Topology* by Jean Monsees, *Navigation* by Philip Stanger, *Plane linkages* by Philip Egeth, *Tricks with numbers* by Lillian Sprung and Virginia Florin, and *The duodecimal system* by Carlton Michelson. Each year the club invites one of the alumni to discuss experiences in the teaching of mathematics. This year the guest speaker was Mrs. Edna H. Young of East Rutherford High School who spoke on *Teaching locus problems with the aid of models.* A joint meeting of mathematics clubs in the state was held in November with representatives from the clubs at New Jersey College for Women, Rutgers University and Upsala College in attendance. Professor Richard Courant of New York University spoke on *Problems of Maxima and Minima.* At a joint meeting with the Science Club, Mr. E. C. Molina of the Bell Telephone Company Laboratories spoke on *Mathematics in the Bell Telephone Industry.* Officers were: President, John Macchi; Vice-President, Jean Monsees; Secretary, Anne Beaumont; Treasurer, Virginia Florin; Librarian, Audrey Vincentz.

Kappa Mu Epsilon, Albion College

Seven meetings were held during the year. *Mathematics in radio* and *Mathematics in aviation* were topics discussed by Ernest Longman and John Telander. Other subjects were presented as follows: *Addition and subtraction of logarithms* by Gerald Allen, *Summation of series* by Mark

Putham, *Theoretical mathematics is practical* by David Lawler, *Pike's early American arithmetics* by Margaret Ingram, *Introduction of zero into the number system* by Webster Sawyer, and *Archimedes* by Helen Shepard.

Pi Mu Epsilon, University of Illinois

Entertaining programs were the aim of the administrative committee during the year. At the opening meeting members heard a humorous paper on the troubles of a newlywed written by Professor A. R. Crathorne entitled *Statistics in the kitchen*. On the evening of election day members took part in a mock election and revised predictions as returns of the voting were received. At a Get-Acquainted Party Dr. Pepper entertained guests with her collection of puzzles and spoke on the subject *A Yank at Oxford*. In February a talk entitled *A night with probability* by Dr. E. R. Blanche was followed by opportunities for all members to compete against gambling devices. Methods of higher algebra were used at another meeting by Professor Harry Levy in solving mathematical puzzles. Professor H. F. Moore of the College of Engineering was guest speaker at the initiation banquet and used as his topic *Wishful thinking and wishful observation*. Officers were: Director, Dr. E. R. Blanche; Adviser, Dr. Echo Pepper; Secretary, DeLos De Tar; Treasurer, Eleanor Ewing.

Mathematics Club, Tennessee Polytechnic Institute

Members of the club participated in a radio program entitled, *Battle between the departments*, consisting of a quiz program in which teams representing the mathematics and engineering departments competed. The mathematics group consisted of Robert Tate, Joseph Lane, Margaret Plumlee and Charles Tabor. Topics discussed at club meetings were: *History of mathematics* by Margaret Plumlee, *Recent trends in arithmetic* by Thurman Webb, *Relation of science to mathematics* by Dr. Moorman, *The number system if we had six fingers* by Dr. Hutchinson, *Simple mathematics problems in electrical work* by Mr. Duncan, *Mathematics in war* by Professor Mattson, *Trisection of angles* by Dr. Hutchinson. At the close of the year the club was accepted as the *Tennessee Alpha Chapter* of *Kappa Mu Epsilon* by the national organization. Officers were: President, K. Walthall; Vice-President, J. Lane; Corresponding Secretary, Dr. R. A. Moorman; Recording Secretary, Margaret Plumlee; Treasurer, W. Fitzgerald.

Pi Mu Epsilon, St. Lawrence University

This chapter held regular meetings throughout the year in conjunction with the local mathematics club, *Alpha Mu Gamma*. Included among the talks were *Actuarial mathematics and the use of statistics in industry* given by Nathan Niles, *Your chance to win* by Roy Jefferey, and *Dimensional analysis* by Walter Boris. Two of the members, Stuart Wadsworth and John Boudiette, demonstrated a mechanical differentiator which they built.* The third annual Pi Mu Epsilon Interscholastic Mathematics Contest was held on May 3, 1941, following the pattern of previous years. Schools competing are members of the Northern New York Interscholastic League which compete annually in football, basketball, and baseball. A cup was awarded to the winning high school team from Potsdam, New York, and medals were given to the three students with the highest scores: John Dooley of Ogdensburg Free Academy, John Turner of Malone Franklin Academy and Ronald Greene of Potsdam High School. Certificates of merit were also awarded to the highest ranking individual of each competing high school. Officers for the year were: President, Constance Weeks; Secretary, Cameron Geraghty; Treasurer, Gerald Bradshaw; Director, Dr. O. K. Bates.

Mathematics Club, Chicago Teachers College

Six meetings were held during the year and the following topics were presented: *The story of the calculus* by George Benyek, *Higher plane curves* by John Conway, *The teaching of mathematics*

* For references used, see this department of the MONTHLY, October 1941, p. 553. Also *Scientific American Supplement* No. 2093, Feb. 12, 1916; *Proc. Royal Society of Edinburgh*, May 1904.

by Dr. Bartky, *Planetary motions* by R. R. Reynolds. The film, *Einstein's Theory of Relativity*, was presented at one program and another meeting was devoted to a description and discussion of the Isograph, slides and a motion picture film being supplied by the Bell Telephone Laboratories and the discussion led by Dr. Mansfield. Officers were: Chairman, H. J. Williams; Vice-Chairman, R. R. Reynolds; Secretary, Asta Einarson; Adviser, Dr. Ralph Mansfield.

Kappa Mu Epsilon, Nebraska State Teachers College at Wayne

Mathematics and amateur radio work was the topic used by Gerald Wright at a fall meeting of the chapter. He based his discussion on his experiences in amateur radio work in which he holds a number of national prizes and he demonstrated his talk with some of his equipment. At another meeting, Mr. Van Bearinger told of his use of the planimeter in measuring aerial photographs while working in a soil conservation office during the summer and this work led to further study of the calculus involved. Officers were: President, J. Ahern; Vice-President, E. Klein; Treasurer, C. Winter; Secretary, Van Bearinger; Faculty Sponsor, Miss E. Marie Hove.

Mathematics Club, Mount Mary College

In addition to attending four meetings of the Intercollegiate Mathematics Association of Milwaukee the club members held two discussion meetings, a joint meeting with the Science Club, a Christmas party and a steak fry. Topics presented were: *Diophantine analysis* by Virginia Altenhofen and *Brocard points* by Marie Hiegel. Officers were: President, Marianne Schueler; Vice-President, Marie Hiegel; Secretary-Treasurer, Margaret Weeks; Adviser, Sister Mary Felice.

Mathematics Club, Butler University

What is mathematics and why study it? was the topic discussed by the club adviser, Mrs. Juna L. Beal at the opening meeting of the year. Later meetings were devoted to talks on *Slide rules and their uses* by Maribelle Foster, *Theory of relativity* by Robert Stump, *History of the calendar* by Jane Gibson, *Some recent discoveries pertaining to the mathematics of the ancient Babylonians* by Helen Caster, *Concepts of the calculus and their development* by Blanchalice Barrett. The final meeting was held at the Goethe Link Observatory at Brooklyn, Indiana, where Dr. Getchell gave a lecture on Astronomy. Officers were: President, Blanchalice Barrett; Vice-President, Helen Caster; Secretary, Maribelle Foster; Treasurer, Robert Stump.

Mathematics-Physics Club, College of Saint Teresa

This organization consisted of 33 members who met bi-monthly to discuss topics in mathematics and physics. The various reports presented by both the faculty members and the students during the year were: *The duo-decimal system, Spectra of molybdenum, Life of Newton, Mathematical magic, The spider lady, Culture in mathematics, Applications of the parallelogram, Telling direction by a watch.* The following films were also presented: *Precisely So, Elgin Tells Time*, and *Geometry in Action*. Officers were: President, Marian Heinen; Vice-President, Mildred Bertrand; Secretary-Treasurer, Margaret Reckers; Faculty Adviser, Sister M. Thomas á Kempis.

Mathematics Club, Boston University

At the first meeting of the year, Professor Bruce gave an illustrated lecture on his travels in India and used for his subject *Algebra's Land of the Dawn*. Mr. C. H. Mergendahl, head of the mathematics department of the Newton Massachusetts high school, was guest speaker at another meeting and illustrated his topic *So what?* with student's reactions to problems. Other subjects presented were: *Natural logarithms* by Mr. Gould, *Magic squares* by Julia Lowe, *In defense of the fourth dimension* by Joseph Rizzo, *Algebraic series* by Joseph Lahage, *History of pi* by Elizabeth Campbell, *Life of Rene Descartes* by Agnes Caneiro, *Brain Teasers* by Francis Scheid. The final program was an Information Please program conducted by Dr. Frye. The members also attended two meetings of the Boston Intercollegiate Mathematics Club Association held at Boston College

and Regis College. Officers were: President, William Gould; Vice-President, Julia Lowe; Secretary, Elizabeth Campbell; Treasurer, Philip Nassisse; Faculty Adviser, Professor Bruce.

Kappa Mu Epsilon, Texas Technological College

In its first year as a chapter of *Kappa Mu Epsilon*, ten program meetings were held. Topics included: Biographies of the mathematicians for whom the chapter officers were named, *Non-Euclidean geometry* by R. K. Wakerling, *The coconut problem* by R. S. Underwood, *Division without a divisor*, by E. R. Heineman, *Number numerology and number theory* by F. W. Sparks, *Projective measurements* by William Wallis, *Curve tracing using Newton's diagram* by Lester LaGrange, *Energy transformations as a source of wealth* by E. A. Hazelwood. Joe R. Foots, first president of the chapter, was appointed part-time instructor in the department of Pure Mathematics of the University of Texas. Lee Michie, another charter member, received his wings and commission as second lieutenant in the Army Air Corps at Stockton, California, on April 25,1941, and sailed from San Francisco on June 3 for the Philippine Islands. Officers were: President Lobatchewsky, William Wallis; Vice-President Agnesi, Aliene May; Secretary Noether, Marie McCrummen; Treasurer Cayley, Rance Jones; Corresponding Secretary Descartes, Mrs. Opal L. Miller; Faculty Sponsor, Dr R. K. Wakerling.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 521. *Proposed by J. R. Musselman, Western Reserve University*

(a) On the sides BC and CA of a triangle ABC, construct externally any two directly similar triangles, CBA_1 and ACB_1. Show that the midpoints of the three segments BC, A_1B_1, CA form a triangle directly similar to the two given triangles.

(b) On BC externally, and on CA internally, construct any two directly similar triangles CBA_1 and CAB_1. Show that the midpoints of AB and A_1B_1 form with C a triangle directly similar to the two given triangles.

E 522. *Proposed by V. Thébault, San Sebastián, Spain*

Find the smallest prime radix for which there exists a perfect cube of the form $abcabc$.

E 523. *Proposed by N. A. Court, University of Oklahoma*

With the vertices of a given orthocentric tetrahedron (T) as centers, spheres are drawn orthogonal to a given sphere (M) concentric with the polar sphere of

(T). Show that the radical planes of (M) with the four spheres considered form a tetrahedron which is orthocentric, and that its orthocenter coincides with that of (T).

E 524. *Proposed by R. V. Heath, Wall St., New York, N.Y.*

Write the numbers 9, 10, 11, 12, 13, 14, 15, 16 in one line. Underneath, place the numbers 1, 2, 3, 4, 5, 6, 7, 8 in such an order that the eight sums and eight differences are sixteen different numbers. In how many ways can this be done?

E 525. *Proposed by Maurice Kraïtchik, New School for Social Research, New York City*

Find parallelepipeds with commensurable edges and diagonals.

SOLUTIONS

An Imperfect Square

E 486 [1941, 555]. *Proposed by J. M. Andreas, Pasadena, California*

A quadrilateral $ABCD$ has a right angle at A. The angles at B and C are bisected by the diagonals BD and CA. Is the quadrilateral necessarily a square?

I. *Solution by Howard Eves, Chattanooga, Tenn.*

We shall answer the question in the negative as follows. Let M be a variable point on a fixed line parallel to a horizontal segment CB. In the triangle MCB, bisect the angles C and B, and let the bisectors meet the respectively opposite sides in A and D. When M is equidistant from C and B, $\angle DAB$ is obtuse; but when M is sufficiently far to the right, this angle is acute. By considerations of continuity it follows that there must be some position of M for which $\angle DAB$ is a right angle. Then $ABCD$ is a non-square quadrilateral satisfying the conditions of the problem.

II. *Solution by E. P. Starke, Rutgers University*

No. If $\angle ABC$ is any obtuse angle, there are two values of the ratio $AB:AC$ such that a quadrilateral may be completed to satisfy the given conditions. To show this, write $\theta = \angle ABD = \frac{1}{2}\angle ABC$, $a = BC$, $c = AB$. By considering projections parallel and perpendicular to AB, one computes without difficulty:

$$AD = c\tan\theta, \quad CD = \sqrt{(a^2 - 2ac + c^2\sec^2\theta)}, \quad AC = \sqrt{(a^2 + c^2 - 2ac\cos 2\theta)}.$$

Equating expressions for the cosines of $\angle ACD$ and $\angle ACB$, one obtains a relation which may be reduced to

$$(a^2 + c^2)(1 - 4\cos^2\theta) + 2ac(1 - 2\cos^2\theta + 4\cos^4\theta) = 0$$

by suppressing a factor $1-\cos^2\theta$. The discriminant of this quadratic in a/c reduces to

$$16\cos^2\theta(1 - \cos^2\theta)(1 - 4\cos^4\theta),$$

which is positive for all θ between 45° and 90°. The statement made above is now obvious.

For a numerical example take $AB=3$, $AD=3\sqrt{2}$, $BC=7+2\sqrt{10}$, $CD=8+\sqrt{10}$, determining $BD=3\sqrt{3}$ and $AC=4\sqrt{2}+4\sqrt{5}$.

Also solved by D. H. Browne, W. B. Clarke, C. W. Emmons, A. K. Waltz and the Clarkson College Mathematical Club. The proposer remarks that the negative answer invalidates a traditional method for obtaining an allegedly square piece of paper from one that is approximately rectangular, by folding and tearing.

The Orthic Triangle

E 487 [1941, 555]. *Proposed by V. Thébault, San Sebastián, Spain*

Prove that if the orthocenter of a triangle is conjugate to the three vertices with regard to the incircle and two of the excircles, respectively, then these three circles touch the respective sides of the orthic triangle, and conversely.

Solution by Howard Eves, Chattanooga, Tenn.

(1) *Lemma*: Given a triangle ABC circumscribed to a conic K, consider points M on AC, N on AB. Then MN touches K if and only if the point of intersection (BM, CN) is conjugate to A.

Let the points of contact of K with AC and AB be S and T, respectively. Suppose MN touches K. Then the simple quadrilateral $BCMN$ is circumscribed to K. Therefore BM and CN are concurrent with ST, the polar of A. Conversely, if BM, CN, ST are concurrent, let the second tangent to K from N cut AC in M'. Then, by the same argument, BM', CN, ST are concurrent; therefore M' coincides with M, and MN is a tangent.

(2) *Theorem*: Given a triangle ABC circumscribed to three conics K_a, K_b, K_c, and given a point H, construct the intersections $L=(AH, BC)$, $M=(BH, CA)$, $N=(CH, AB)$. Then the sides of the triangle LMN touch the respective conics K_a, K_b, K_c if and only if H is conjugate to A, B, C with regard to these respective conics.

This follows at once from the lemma.

(3) The given problem is a highly specialized case of (2).

The proposer remarks that, for such a triangle, the diameter of the circumcircle is equal to the radius of the third excircle.

Differences of Factorials

E 488 [1941, 556]. *Proposed by D. H. Browne, Buffalo, N.Y.*

The factorial $u_k=k!$, "sub-factorial" $v_k=k!\sum_0^k(-1)^r/r!$, and "super-factorial" $w_k=k!\sum_0^k 1/r!$, may be defined by the recurrence formulas

$$u_0 = v_0 = w_0 = 1, \qquad u_k = ku_{k-1}, \qquad v_k = kv_{k-1} + (-1)^k, \qquad w_k = kw_{k-1} + 1.$$

Show that $\Delta^n w_0 = u_n$, $\Delta^n u_0 = v_n$.

Solution by R. D. Specht, University of Florida

We have

$$\Delta^n w_0 = (E-1)^n w_0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r} w_{n-r}$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} (n-r)! \sum_{s=0}^{n-r} \frac{1}{s!} = n! F_n$$

where

$$F_n = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-1)^r}{r!s!}.$$

But

$$\Delta F_n = \sum_{r=0}^{n+1} \frac{(-1)^r}{r!(n+1-r)!} = \frac{(1-1)^{n+1}}{(n+1)!} = 0$$

and $F_0 = 1$. Therefore $F_n = 1$, and $\Delta^n w_0 = n! = u_n$.

Also

$$\Delta^n u_0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r} u_{n-r} = n! \sum_{r=0}^{n} \frac{(-1)^r}{r!} = v_n.$$

Editorial Note. The first part may be proved more simply by observing that

$$\Delta^n w_0 - n\Delta^{n-1} w_0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r} w_{n-r} - n \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} w_{n-r-1}$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} \{ w_{n-r} - (n-r) w_{n-r-1} \}$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0.$$

Sections of a Prismatoid

E 489 [1941, 556]. *Proposed by Howard Eves, Chattanooga, Tenn.*

Let A_0, A_m, A_h be the areas of the lower base, midsection, and upper base of a pristmatoid. If $A_h = A_0$, prove that

(1) sections equidistant from the midsection are equal in area;

(2) the midsection bisects the volume of the prismatoid;

(3) if $A_m = A_0$, all sections have the same area;

(4) if $A_m \neq A_0$, A_m is the maximum or minimum section.

Solution by the Proposer

Let S be a solid figure extending between two parallel planes. Let h be the distance between the planes, and let A_x be the area of the section at distance x

from one of the planes (the "lower base"). Suppose S is such that A_x is a quadratic function of x. (The prismatoid, along with many other figures, is such a solid.) Now we have

$$A_x = rx^2 + sx + t, \tag{i}$$

where r, s, t are constants to be determined. Putting $x=0$, h, $h/2$ in turn, and solving for r, s, t, we find

$$r = (2A_0 - 4A_m + 2A_h)/h^2, \qquad s = (4A_m - A_h - 3A_0)/h, \qquad t = A_0.$$

Substituting in (i) we get

$$h^2A_x = (2A_0 - 4A_m + 2A_h)x^2 + (4A_m - A_h - 3A_0)hx + h^2A_0.$$

When $A_h = A_0$, this reduces to

$$h^2A_x = (2x - h)^2A_0 - 4x(x - h)A_m. \tag{ii}$$

From (ii) it is clear that $A_x = A_{h-x}$, which is part (1) of the problem. Cavalieri's Theorem makes part (2) an immediate corollary of this. Putting $A_m = A_0$ in (ii), we see that $A_x = A_0$, which is part (3). Differentiating the right member of (ii) with respect to x, we see that A_x is a maximum or minimum when $x = h/2$; this establishes part (4).

Derangements

E 490 [1941, 556]. *Proposed by J. F. Kenney, University of Wisconsin at Milwaukee*

In a gambling game, a player is permitted to deal ten cards from a bridge deck (which has been thoroughly shuffled) and wins if, at any stage of the dealing, the number on a card is the same as the number of cards dealt. (Face cards are assigned the number 0.) Find the probability that the dealer will win.

I. *Solution by R. W. Wagner, Oberlin College*

Let N_i denote the number of ways of arranging the deck so that the first i cards are respectively an ace, a two, and so on. Then $N_i = 4^i(52-i)!$. By Proposition XIV of Whitworth, *Choice and Chance*, p. 68, we deduce that the number of ways of arranging the deck so that no one of the first ten cards has a number equal to its position is

$$\sum_{i=0}^{10} (-1)^i \binom{10}{i} N_i = \sum_{i=0}^{10} (-4)^i \binom{10}{i} (52 - i)!.$$

The total number of ways of arranging the deck is 52!. Thus the probability of winning is

$$-\sum_{i=1}^{10} (-4)^i \binom{10}{i} (52 - i)!/52! = .54815 \cdots .$$

II. *Solution by Irving Kaplansky, Harvard University*

This is a special case of the problem treated in my note *On a generalization of the "Problème des Rencontres"* [1939, 159]. Putting $n=13$, $a_i=4$, $p_1=\cdots=p_{10}=1$, $p_{11}=p_{12}=p_{13}=0$ in the formula obtained there, we get

$$E^{12}(E^4-4E^3)^{10}0!=E^{42}(E-4)^{10}0!=\sum_{i=0}^{10}(-4)^i\binom{10}{i}(52-i)!$$

for the number of arrangements.

III. *Experimental check by D. H. Browne, Buffalo, N. Y.*

By a trial of several hundred deals, using a single close ruff and a lift cut between each deal and the next, I get an average of 54.5%.

Also solved by E. P. Starke and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4036. *Proposed by L. A. Santaló, Rosario, Argentina*

Let C_1 be an oval with a continuously varying radius of curvature R; at each point of C_1 a normal of length R is drawn exteriorly giving points of a second curve C_2 (which may not be convex); and let A be the area enclosed between the two curves. From a chosen fixed point a vector is drawn parallel to the normal at a point of C_1 and of length R for that point, thus giving as the point varies on C_1 a curve C_3 having the area A_3 and length L_3. If L_2 is the length of C_2 and A_1 is the area of C_1, show that

$$\text{(a)}\quad A=3A_3; \qquad \text{(b)}\quad L_2L_3\geqq 8\pi A_1;$$

the equality in (b) is true only when C_1 is a circle.

4037. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Integrate

$$(x^{n+1}+y^n)y'-x^ny=0;$$

calculate and examine the radius of curvature of the integral curves at the origin.

4038. *Proposed by V. Thébault, San Sebastián, Spain*

The point M is chosen arbitrarily on a bisector of angle A of the triangle ABC, and let M' be its isogonal conjugate with respect to ABC. Show that the two circles each through M and M' and tangent to the side BC are tangent also to the circumcircle of ABC.

4039. *Proposed by N. A. Court, University of Oklahoma*

The circumcenter of a tetrahedron (T) and any point M are isogonal conjugates with respect to the tetrahedron formed by the centers of the four spheres passing through M and the circumcircles of the faces of (T).

Correction of editorial errors in 3994 [1941, 273]. *Proposed by C. E. Springer, University of Oklahoma*

If

$$a_{11} = a_{22} = a_{33} = \sum_j \binom{K}{j} (n-1)^{K-j}, \qquad j \equiv 0 \pmod 3;$$

$$a_{12} = a_{23} = a_{31} = \sum_j \binom{K}{j} (n-1)^{K-j}, \qquad j \equiv 1 \pmod 3;$$

$$a_{13} = a_{32} = a_{21} = \sum_j \binom{K}{j} (n-1)^{K-j}, \qquad j \equiv 2 \pmod 3;$$

show that the determinant

$$|a_{ij}| = [(n-1)^3 + 1]^K.$$

SOLUTIONS

Symmetric Functions

3980 [1941, 69]. *Proposed by Esther Szekeres, Budapest, Hungary*

The symmetric polynomials $y_1, y_2, \cdots, y_n$ in the variables $x_1, x_2, \cdots, x_n$ are of the degrees indicated by the subscripts, and are algebraically independent. If $f(x_1, x_2, \cdots, x_n)$ is any given polynomial symmetric in the x's, show that it can be expressed as a polynomial in the y's.

Solution by the Proposer

Since the symmetric polynomial $f(x_1, x_2, \cdots, x_n)$ can be expressed as a polynomial in terms of the elementary symmetric functions $\sigma_1, \sigma_2, \cdots, \sigma_n$, it suffices to show that each σ_i can be expressed as a polynomial in the y_i's.

We may write

$$y_k = c_k\sigma_k + g_k(\sigma_1, \sigma_2, \cdots, \sigma_{k-1}),$$

where the second term in the right member is a polynomial in the indicated arguments whose terms are of weights not exceeding k, or in other words it is the sum of terms such as

$$\sigma_1^{t_1}\sigma_2^{t_2}\cdots\sigma_{k-1}^{t_{k-1}}, \qquad t_1+2t_2+\cdots+(k-1)t_{k-1}\leqq k,$$

multiplied by a constant coefficient. We show first that no c_k is zero, If any one is zero let c_k be the first one to be zero. Then we have

$$\begin{aligned}
y_1 &= c_1\sigma_1 + c_0\\
y_2 &= c_2\sigma_2 + g_2(\sigma_1)\\
&\cdots\\
y_{k-1} &= c_{k-1}\sigma_{k-1} + g_{k-1}(\sigma_1, \sigma_2, \cdots, \sigma_{k-2})\\
y_k &= g_k(\sigma_1, \sigma_2, \cdots, \sigma_{k-1}).
\end{aligned}$$

These equations may be solved in turn for $\sigma_1, \sigma_2, \cdots, \sigma_{k-1}$ so that $\sigma_i (i\leqq k-1)$ is a polynomial in $y_1, y_2, \cdots, y_i$; and then inserting these values in the last equation, we shall have

$$y_k = g(y_1, y_2, \cdots, y_{k-1}),$$

where the right member is a polynomial in the indicated arguments, But this is contrary to the hypothesis. Hence no c_k is zero, and each σ_i can be expressed as a polynomial in $y_1, y_2, \cdots, y_i$. This concludes the proof.

Solved also by C. M. Stein. A solution similar to the one by the proposer was received from Li Ming-hsien after the preparation of the above for printing.

Spheres Associated with a Tetrahedron

3982 [1941, 70]. *Proposed by V. Thébault, San Sebastián, Spain*

The vertices of the tetrahedron $ABCD$ are centers of spheres the squares of whose radii are equal respectively to one-third of the sum of the squares of the edges through the considered vertex. Show that the sphere orthogonal to the four spheres is concentric with the twelve point sphere of $ABCD$.

Note. See N. A. Court, *Modern Pure Solid Geometry*, p. 250, for the twelve point sphere.

3983 [1941, 70]. *Proposed by V. Thébault, San Sebastián, Spain*

The vertices of the tetrahedron $ABCD$ are centers of spheres the squares of whose radii are equal respectively to k times the sum of the squares of the edges of the face opposite to the vertex considered, and they are also centers of spheres the squares of whose radii are equal respectively to k times the sum of the squares of the edges through the considered vertex. Let ω_1 and ω_2 be the centers of the spheres, radii R_1 and R_2, orthogonal respectively to the two sets of four spheres, G the centroid, and O the circumcenter of $ABCD$. (1) Show that the points O, G, ω_1, ω_2 are collinear and determine their relative positions. (2) Show that $(R_1^2-R_2^2)/k$ remains constant when k varies.

Solution by J. W. Peters, University of Illinois

Since the first problem is a special case of part of the second, let us consider the second problem first.

Let $\overline{A}_i$, $(i=1, 2, 3, 4)$, be the position vectors of the vertices of the tetrahedron $ABCD$. There will be no loss in generality if the origin is taken as the circumcenter, O, and if the unit of measurement is so chosen that the radius of the circumsphere is 1. The vectors $\overline{A}_i$ are thus unit vectors. Let $\overline{S}=\sum_{i=1}^{4}\overline{A}_i$. Then from the reference to Court we learn that the centroid is $\overline{S}/4$, the Monge point is $\overline{S}/2$, and the center of the twelve point sphere is $\overline{S}/3$.

Let the first set of spheres mentioned in problem 3983 be given by

$$(\overline{X}-\overline{A}_i, \overline{X}-\overline{A}_i)=r_i^2, \tag{1}$$

where

$$r_i^2 = k[(\overline{A}_j-\overline{A}_m, \overline{A}_j-\overline{A}_m)+(\overline{A}_m-\overline{A}_n, \overline{A}_m-\overline{A}_n)+(\overline{A}_n-\overline{A}_j, \overline{A}_n-\overline{A}_j)],$$
$$r_i^2 = 2k[3-(\overline{A}_j, \overline{A}_m)-(\overline{A}_m, \overline{A}_n)-(\overline{A}_n, \overline{A}_j)],$$

where k is a real scalar and $i, j, m, n, =1, 2, 3, 4$; $i\neq j\neq m\neq n$. The parentheses represent scalar products of the enclosed vectors.

Since $(\overline{X}-\bar{\omega}_1, \overline{X}-\bar{\omega}_1)=R_1^2$ is orthogonal to the spheres (1), we have $(\overline{A}_i-\bar{\omega}_1, \overline{A}_i-\bar{\omega}_1)-R_1^2-r_1^2=0$, which reduces to

$$1-2(\overline{A}_i, \bar{\omega}_1)+(\bar{\omega}_1, \bar{\omega}_1)-R_1^2-r_i^2=0. \tag{2}$$

Eliminating R_1^2 from this set of four equations, we obtain three independent equations of the form

$$2(\overline{A}_i-\overline{A}_j, \bar{\omega}_1)+r_i^2-r_j^2=0. \tag{3}$$

On replacing the r_i^2 by their values in terms of k and A_i, this set of equations reduces to a set of three equations of the form

$$(\overline{A}_i-\overline{A}_j, \bar{\omega}_1)+k(\overline{A}_i-\overline{A}_j, \overline{S})=0. \tag{4}$$

If the four points $ABCD$ do not lie in a plane, the equations (4) will have the solution $\bar{\omega}_1=-k\overline{S}$. With this value for $\bar{\omega}_1$ and any one of equation (2), we find

$$R_1^2=(1-2k)^2+2k(1+k)\cdot\sum^{6}(\overline{A}_i, \overline{A}_j), \qquad i, j=1, 2, 3, 4;\ i\neq j. \tag{5}$$

The second set of four spheres with centers at the vertices $ABCD$ are given by the equations

$$(\overline{X}-\overline{A}_i, \overline{X}-\overline{A}_i)=s_i^2, \tag{6}$$

where

$$s_i^2 = k[(\overline{A}_i-\overline{A}_j, \overline{A}_i-\overline{A}_j)+(\overline{A}_i-\overline{A}_m, \overline{A}_i-\overline{A}_m)+(\overline{A}_i-\overline{A}_n, \overline{A}_i-\overline{A}_n)]$$

or

$$s_i^2 = 2k[4-(\overline{A}_i, \overline{S})].$$

Since $(\overline{X}-\bar{\omega}_2,\ \overline{X}-\bar{\omega}_2)=R_2^2$ is orthogonal to the spheres (6), it follows that

$$1-2(\overline{A}_i,\ \bar{\omega}_2)+(\bar{\omega}_2,\ \bar{\omega}_2)-R_2^2-s_i^2=0. \tag{7}$$

Eliminating R_2^2 from these four equations, one obtains a set of three independent equations of the form

$$2(\overline{A}_i-\overline{A}_j,\ \bar{\omega}_2)+s_i^2-s_j^2=0. \tag{8}$$

On replacing the s_i^2 by their values in terms of k and A_i, this set of equations reduces to

$$(\overline{A}_i-\overline{A}_j,\ \bar{\omega}_2)-k(\overline{A}_i-\overline{A}_j,\ \overline{S})=0. \tag{9}$$

If the four points are not in a plane, the three equations (9) will have the solution $\bar{\omega}_2=k\overline{S}$. With this value for $\bar{\omega}_2$ and any one of the equations (7), it is found that

$$R_2^2=1-8k+4k^2+2k^2\cdot\sum^{6}(\overline{A}_i,\ \overline{A}_j). \tag{10}$$

It is evident that $\bar{\omega}_1$ and $\bar{\omega}_2$ lie on the line OG and that O is the bisector of the segment $\bar{\omega}_1\bar{\omega}_2$. Furthermore it is easily seen that $(R_1^2-R_2^2)/k=4+2\sum(\overline{A}_i,\ \overline{A}_j)$ and this ratio, being independent of k, remains constant when k varies.

For problem 3982, set $k=1/3$ in equations (6), (7), (8), and (9). Then $\bar{\omega}_2=\overline{S}/3$ and this is the center of the twelve point sphere.

Solved also by the proposer.

Isogonal Conjugates

3984 [1941, 152]. *Proposed by R. Goormaghtigh, Bruges, Belgium*

The two points P, Q are symmetric as to the circumcenter of a triangle, the isogonal conjugates of P, Q are P', Q', and M is the midpoint of $P'Q'$; prove that $PQ\cdot P'Q'=4R\cdot HM$, where H is the orthocenter and R is the circumradius of the triangle.

Solution by Li Ou, Yenching University, Peiping, China

Let the vertices of the triangle $A_1A_2A_3$ be the turns t_i satisfying $t^3-\sigma_1t^2+\sigma_2t-\sigma_3=0$; and let the point P be the complex number z. Since P and Q are symmetric as to the circumcenter of the triangle, the complex number of Q will be $-z$ Then it may be shown* that the complex number of P', the isogonal conjugate of P, is

$$P'\equiv\frac{\bar{z}^2\sigma_3-\bar{z}\sigma_2+\sigma_1-z}{1-z\bar{z}},$$

where $\bar{z}$ is the conjugate of z. Similarly the isogonal conjugate of Q will be

* J. H. Weaver, On Isogonal Points, this MONTHLY, vol. 42, 1935, p. 496.

$$Q' \equiv \frac{\bar{z}^2\sigma_3 + \bar{z}\sigma_2 + \sigma_1 + z}{1 - z\bar{z}}.$$

Hence we have the lengths

$$\overline{PQ} = 2\,|\,z\,|, \qquad \overline{P'Q'} = \frac{2\,|\,\bar{z}\sigma_2 + z\,|}{|\,1 - z\bar{z}\,|}.$$

Thus, their product

$$\overline{PQ}\cdot\overline{P'Q'} = \frac{4\,|\,z\bar{z}\sigma_2 + z^2\,|}{|\,1 - z\bar{z}\,|}.$$

Now the complex numbers of M and H are easily shown to be

$$M \equiv \frac{\bar{z}^2\sigma_3 + \sigma_1}{1 - z\bar{z}}, \qquad H \equiv \sigma_1$$

respectively; and the length

$$\overline{HM} = \frac{|\,\sigma_3\,|\,|\,z\bar{z}\sigma_2 + \bar{z}^2\,|}{|\,1 - z\bar{z}\,|} = \frac{|\,z\bar{z}\sigma_2 + z^2\,|}{|\,1 - z\bar{z}\,|},$$

since $|\sigma_3| = 1$. Hence $\overline{PQ}\cdot\overline{P'Q'} = 4\cdot\overline{HM}$. We note that here we take $R = 1$.

Solved also by J. W. Clawson, J. H. Weaver, and the proposer, each using complex coordinates.

An Operational Identity

3985 [1941, 152]. *Proposed by Charles M. Stein, New York, N. Y.*

Prove that if $f(x)$ is analytic within and on a circle C having x as center and containing y, then

$$\cdots\left(1 + \frac{y-x}{n}D\right)\cdots\left(1 + \frac{y-x}{2}D\right)\left(1 + \frac{y-x}{1}D\right)f(x) = f(y),$$

where $D = d/dx$.

Solution by Kwan Chao-Chih, Yenching University, Peiping, China

Since $f(x)$ is analytic in C, all its derivatives exist. We observe

$$\left(1 + \frac{y-x}{1}D\right)f(x) = f(x) + \frac{y-x}{1}f'(x) \equiv \sum_{\gamma=0}^{1}\frac{(y-x)^\gamma}{\gamma!}f^{(\gamma)}(x).$$

We shall write

$$\left(1 + \frac{y-x}{n-1}D\right)\cdots\left(1 + \frac{y-x}{2}D\right)\left(1 + \frac{y-x}{1}D\right)f(x) = \sum_{\gamma=0}^{n-1}\frac{(y-x)^\gamma}{\gamma!}f^{(\gamma)}(x), \tag{1}$$

and prove it by induction for any n. For,

$$\left(1+\frac{y-x}{n}D\right)\cdots\left(1+\frac{y-x}{2}D\right)\left(1+\frac{y-x}{1}D\right)f(x)$$
$$=\sum_{\gamma=0}^{n-1}\frac{(y-x)^\gamma}{\gamma!}f^{(\gamma)}(x)+\frac{(y-x)}{n}\left[\sum_{\gamma=0}^{n-1}\frac{(y-x)^\gamma}{\gamma!}f^{(\gamma+1)}(x)\right.$$
$$\left.-\sum_{\gamma=1}^{n-1}\frac{(y-x)^{\gamma-1}}{(\gamma-1)!}f^{(\gamma)}(x)\right]$$
$$=\sum_{\gamma=0}^{n}\frac{(y-x)^\gamma}{\gamma!}f^{(\gamma)}(x).$$

When $n\to\infty$, the right member of (1) is Taylor's expansion of $f(x)$ at y, and thus represents $f(y)$ since the circle C contains y.

Solved also by Huang K'un and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to C. O. Oakley, Haverford College, Haverford, Pennsylvania.

The National Council of Teachers of Mathematics will hold its next annual summer meeting jointly with the Department of Secondary Education of the National Education Association on June 29 and 30, 1942, at Denver, Colorado.

The twenty-fifth anniversary of Dr. H. E. Hawkes, professor of mathematics, as dean of Columbia College was celebrated at a dinner given in his honor on April 16, 1942.

Dr. Archie Blake, associate mathematician of the U. S. Coast and Geodetic Survey, has been transferred to the National Inventors Council at Washington.

C. A. Bridger, statistician for the Idaho Department of Public Health, has been given a civil service appointment as junior metallurgist with the Bureau of Mines at Salt Lake City.

Assistant Professor Marguerite D. Darkow of Hunter College has been promoted to an associate professorship.

Assistant Professor B. E. Gatewood of Louisiana Polytechnic Institute is on leave of absence to take a position at the MacDonnell Aircraft Corporation at St. Louis.

Assistant Professor C. W. Hook of Georgia School of Technology, a member of the Naval Reserve, has reported to the Naval Academy for duty as an instructor in mathematics.

At Smith College, Associate Professors N. H. McCoy and Deane Montgomery have been promoted to professorships.

R. M. Pinkerton, associate physicist at Langley Field, has been appointed associate professor in the Aeronautical Engineering Department at Agricultural and Mechanical College of Texas.

Professor D. W. Pugsley of Berea College has succeeded Professor W. R. Hutcherson as head of the department of mathematics, Professor Hutcherson having been granted leave of absence on a defense mechanics fellowship at Brown University.

Assistant Professor E. J. Purcell has been promoted to an associate professorship at the University of Arizona.

Professor Suzan R. Benedict of Smith College died of a heart attack on April 8, 1942. She was a charter member of the Mathematical Association and had retired from teaching last January after thirty-five years teaching at Smith College.

Dr. Robert Henderson, vice president and actuary (retired) of the Equitable Life Assurance Society, died February 16, 1942. He was a charter member of the Mathematical Association of America and was for a long term of years a trustee of the American Mathematical Society.

Assistant Professor Elizabeth E. Knight of State Teachers College of Milwaukee died on March 26, 1942. She had been a member of the Mathematical Association since 1925.

Dr. W. W. Landis, for forty-seven years professor of mathematics at Dickinson College, died on April 8, 1942, at the age of seventy-three years. He was a charter member of the Mathematical Association.

Professor J. H. Weaver of Ohio State University was instantly killed by a train near his home on April 7, 1942. He was a charter member of the Mathematical Association and had assisted on the staff of the MONTHLY for a number of years.

SUMMER COURSES

The following courses in mathematics are announced for the summer of 1942:

University of Chicago. In view of the national emergency the Summer Quarter has been lengthened to twelve weeks, extending from *June 22 to September 12, 1942.* However, most of the following courses, with the exception of trigonometry, algebra and possibly calculus, will be completed by *August 29,* for the benefit of those teachers who must return to their positions. In addition to courses in trigonometry, algebra, analytic geometry, calculus and differential equations, the following will be offered: By Professor Bliss (with the cooperation of Professor Graves, Hestenes, Reid, Radó and Smiley): Seminar on the calculus of variations. By Professor Barnard: Lattice theory. By Professor Graves: Func-

tions of real variables. By Professor Hartung: Advanced problems in teaching mathematics in the secondary school and the junior college. By Professor Hestenes: Calculus of variations. By Professor Lane: Metric differential geometry, Projective differential geometry of hyperspace. By Professor Radó: Subharmonic functions. By Professor Reid: Fourier series and Bessel functions, Exterior ballistics. By Dr. Schilling: Introduction to algebraic theories, Galois theory.

Iowa State College. Statistics will be emphasized in the first summer session, *June 8 to July 15.* Courses in the mathematical theory of statistics and its applications will be offered; the design of experiments and of other sampling investigations important in the present emergency will receive special consideration.

University of North Carolina. In addition to the regular courses through the calculus, the following will be offered: *First Term, June 11 to July 21.* By Professor Henderson: Advanced algebra, Theory of equations. By Professor Browne: Theory of numbers. By Professor Hoyle: Advanced calculus. *Second Term, July 22 to August 28.* By Professor Lasley: Projective geometry. By Professor Mackie: Differential equations. By Professor Hill: History of mathematics.

University of Pennsylvania. In addition to the usual elementary courses, the following advanced work will be offered: By Professor Rademacher: Partial differential equations of mathematical physics, Elliptic functions. By Professor Shohat: Theory and practice of approximations. By Professor Clarkson: Navigation.

THE WILLIAM LOWELL PUTNAM COMPETITION

The following are the results of the fifth annual William Lowell Putnam Mathematical Competition held March 7, 1942.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of the University of Toronto. The members of the team were K. S. Hoyle, H. V. Lyons, M. A. Preston; to each of these is awarded a prize of forty dollars.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Yale University. The members of the team were F. H. Brownell, 3rd, A. M. Gleason, A. E. Roberts, Jr.; to each of these is awarded a prize of thirty dollars.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology. The members of the team were E. D. Calabi, W. S. Loud, G. P. Wachtell; to each of these is awarded a prize of twenty dollars.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the College of the City of New York. The members of the team were Herman Chernoff, Harvey Cohn, Edward Gordon; to each of these is awarded a prize of ten dollars.

The five persons ranking highest in the examination, names in alphabetical order, were Harvey Cohn, College of the City of New York; A. M. Gleason, Yale University; W. S. Loud, Massachusetts Institute of Technology; H. V. Lyons, University of Toronto; M. A. Preston, University of Toronto. Each of these will receive a prize of fifty dollars.

The following teams won honorable mention: Department of Mathematics, Cooper Union Institute of Technology, the members of the team being Harold Grad, M. S. Klamkin, Kenneth Robinson; Department of Mathematics, Harvard University, the members of the team being R. M. Bloch, L. S. Shapley, J. A. Zilber; Department of Mathematics, New York University, the members of the team being Melvin Lax, Harold Lewis, Henry Shenker; Department of Mathematics, Swarthmore College, the members of the team being N. V. Hannay, W. H. Mills, M. S. Raff. This list is alphabetical.

Five individuals are given honorable mention, the names listed in alphabetical order being: E. D. Calabi, Massachusetts Institute of Technology; C. P. Gadsden, Tulane University; K. S. Hoyle, University of Toronto; Melvin Lax, New York University; W. H. Mills, Swarthmore College.

A total of one hundred fourteen undergraduate students representing thirty-one institutions took part in the competition.

W. D. CAIRNS, *Secretary-Treasurer*

EXAMINATION QUESTIONS FOR THE FIFTH WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION, MARCH 7, 1942

MORNING SESSION: 9:00 A.M. to 12:00 NOON. (*Answer the questions in any order and by any method. Show all your work in logical sequence, and indicate your answers clearly. No tables or other books may be used.*)

1. A square of side $2a$, lying always in the first quadrant of the XY plane, moves so that two consecutive vertices are always on the X- and Y-axes respectively. Find the locus of the mid-point of the square.

2. The polynomial $f(x)$ is divided by $(x-a)^2\,(x-b)$, where $a \neq b$, derive a formula for the remainder.

3. Is the following series convergent or divergent

$$1+\frac{1}{2}\cdot\frac{19}{7}+\frac{2!}{3^2}\left(\frac{19}{7}\right)^2+\frac{3!}{4^3}\left(\frac{19}{7}\right)^3+\frac{4!}{5^4}\left(\frac{19}{7}\right)^4+\cdots?$$

4. Find the orthogonal trajectories of the family of conics $(x+2y)^2=a(x+y)$. At what angle do the curves of one family cut the curves of the other family at the origin?

5. A circle of radius a is revolved through 180° about a line in its plane, distant b from the center of the circle, where $b>a$. For what value of the ratio b/a does the center of gravity of the solid thus generated lie on the surface of the solid?

6. Any circle in the XY (horizontal) plane is "represented" by a point on the vertical line through the center of the circle, and at a distance "above" the plane of the circle equal to the radius of the circle.

Show that the locus of the representations of all the circles which cut a fixed circle at a constant angle is a (portion of a) one-sheeted hyperboloid.

By consideration of suitable families of circles in the plane, demonstrate the existence of two families of rulings on the hyperboloid.

AFTERNOON SESSION: 2:00 P.M. TO 5:00 P.M. (*Answer the questions in any order and by any method. Show all your work in logical sequence, and indicate your answers clearly. No tables or other books may be used.*)

7. A square of side $2a$, lying always in the first quadrant of the XY plane, moves so that two consecutive vertices are always on the X- and Y-axes respectively. Prove that a point within or on the boundary of the square will in general describe a (portion of a) conic. For what points of the square does this locus degenerate?

8. For the family of parabolas

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a,$$

(a) Find the locus of vertices.
(b) Find the envelope.
(c) Sketch the envelope and two typical curves of the family.

9. Given

$$x = \phi(u, v)$$
$$y = \psi(u, v)$$

where ϕ and ψ are solutions of the partial differential equation

$$\frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial v} - \frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial u} = 1. \tag{1}$$

By assuming that x and v are the independent variables, show that (1) may be transformed to

$$\frac{\partial y}{\partial v} = \frac{\partial u}{\partial x}. \tag{2}$$

Integrate (2), and show how this effects in general the solution of (1). What other solutions does (1) possess?

10. A particle moves under a central force inversely proportional to the kth power of the distance. If the particle describes a circle (the central force proceeding from a point *on the circumference of the circle*), find k.

11. Sketch the curve

$$y = \frac{x}{1 + x^6 \sin^2 x},$$

and show that

$$\int_0^\infty \frac{x\,dx}{1 + x^6 \sin^2 x}$$

exists.

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The examination lists were formulated by Professors B. H. Brown and Marie Litzinger.

Note. Chairmen of mathematics departments may obtain copies of the examination questions for the Putnam Competition for 1938, for 1939, for 1940, for 1941, and for 1942 by a post card request to Professor W. D. Cairns, 97 Elm Street, Oberlin, Ohio.

THE NEW SECRETARY-TREASURER

At the meeting of the Board of Governors held at Chicago on September 1, 1941, Professor William D. Cairns, Secretary-Treasurer of the Association since its organization in 1915, announced his wish and purpose to retire from that position at the end of his present term of office, January 1, 1943.

On behalf of the officers and Board of Governors of the Mathematical Association of America, I am very happy to announce the election of Professor Walter B. Carver of Cornell University as Secretary-Treasurer for a five-year term beginning in January, 1943. As a result of this appointment Professor Carver will continue his long service to the Association, which includes five years as Editor-in-Chief, two years as President, and further service on the Finance Committee. The Association is most fortunate to have its affairs in such competent hands during the important years just ahead.

R. W. BRINK, *President*

PRE-INDUCTION TRAINING

Bulletin 23 on *Higher Education and National Defense*, published by the American Council on Education, is devoted to the subject "Pre-induction Training Needs on the College Level for Enlisted Men in the Armed Forces." Since any pre-training needed by enlisted men obviously is at least equally important for officers, we may look upon the following quotations from this bulletin as applying to officers as well as to enlisted men. In the preparation of the bulletin, an analysis of scholastic needs was made through the medium of consultations with the officers in charge of training in the various branches of the armed forces. The following quotations from Bulletin 23 are of interest as relating to mathematics or to subjects demanding a mathematical basis.

"Their [the officers'] statements of suggested educational background were carefully digested for this report, and their opinions should be given serious thought even though there seems to be only a new emphasis and not new subject-matter material in this list of studies. Two things should be kept in

mind, however, to interpret properly the suggested studies by the officers. First, they had been asked to consider students who would be in college, short of graduation. Second, they naturally were thinking, in the main, of background material which would shorten, for the armed forces, the training period for specific jobs, They, therefore, strongly recommend the following studies:

Mathematics: arithmetic, algebra, geometry, trigonometry, and some calculus, functions, graphs, and some surveying.

Physics: standard course with special emphasis on the following: electricity and magnetism, hydraulics, mechanics, heat, light, sound, force and motion, optics, principles of internal combustion engine, telephone, telegraph and radio communication, Morse code and International code.

Chemistry: general course to include principles of explosives.

Physical Geography: map interpretation, especially topographic maps and aerial photography, winds and weather.

Astronomy: as it relates to air and marine navigation, descriptive.

W. L. HART

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fifth Summer Meeting, Poughkeepsie, N. Y., September 7–9, 1942.
Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Decatur, May 8–9, 1942
INDIANA, Notre Dame, April 9–10, 1943
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Ashland, Va., May 2, 1942
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Northfield, May 9, 1942
MISSOURI, fall, 1942
NEBRASKA, Omaha, May 9, 1942
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, fall, 1942
WISCONSIN, Milwaukee, May 7, 1943

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